

PART V

Boundary Element Method

BOUNDARY ELEMENT METHOD I

- **Boundary Element Method** — alternative approach to solution of boundary value problems
- Motivation — consider the following initial value problem

$$\frac{du(t)}{dt} = f(u(t), t), \quad u(0) = u_0$$

This problem can be alternatively restated as

$$u(t) = u_0 + \int_0^t f(u(\tau), \tau) d\tau$$

- When solving the problem numerically, either of the forms can be taken as the point of departure
- The Boundary Element Method relies on an **integral** representation of the solution of an elliptic problem

BOUNDARY ELEMENT METHOD II

- Let D be a bounded, open domain in \mathbb{R}^2 with a boundary S . At any point $P \in S$ the inner unit normal vector is denoted by \mathbf{n}_P ; to fix attention, we will consider the following two boundary-value problems:

- The interior Dirichlet problem — given $f \in C(S)$, find $u \in C(\bar{D}) \cap C^2(D)$ such that

$$\begin{cases} \Delta u(P) = 0, & P \in D \\ u(P) = f(P), & P \in S \end{cases}$$

- The interior Neumann problem — given $f \in C(S)$, find $u \in C(\bar{D}) \cap C^2(D)$ such that

$$\begin{cases} \Delta u(P) = 0, & P \in D \\ \frac{\partial u(P)}{\partial \mathbf{n}_P} = f(P), & P \in S \end{cases}$$

- Generalizations to other boundary conditions, 3D domains and more complicated linear equations are possible.

BOUNDARY ELEMENT METHOD III

- Given $u, w \in C^2(\bar{D})$, consider Green's identity (obtained from the divergence theorem)

$$\int_D (u\Delta w - w\Delta u) d\Omega = \oint_S \left(w \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial w}{\partial \mathbf{n}} \right) dS$$

- Assume that u is the solution of the Laplace equation and $w = \frac{1}{2\pi} \ln |A - Q|$, i.e., the fundamental solution of the Laplace equation in an unbounded domain ($|A - Q|$ denotes the Euclidean norm of the vector $A - Q$); the above relation can be converted into

$$u(A) = \frac{1}{2\pi} \oint_S \left[\frac{\partial u(Q)}{\partial \mathbf{n}_Q} \ln |A - Q| - u(Q) \frac{\partial}{\partial \mathbf{n}_Q} \ln |A - Q| \right] dS_Q, \quad A \in D$$

- Thus, the solution of the Laplace equation can be represented as an integral involving **both** the **Dirichlet** and **Neumann** boundary conditions; note that only one of them is available.

BOUNDARY ELEMENT METHOD IV

- Thus, if
 - $u(Q)$ is given, need to determine $\frac{\partial u(Q)}{\partial \mathbf{n}_Q}$,
 - $\frac{\partial u(Q)}{\partial \mathbf{n}_Q}$ is given, need to determine $u(Q)$
- When $A \in D$ approaches $P \in S$ the following limits can be calculated in the integral equation

$$\lim_{A \rightarrow P} \int_S \frac{\partial u(Q)}{\partial \mathbf{n}_Q} \ln |A - Q| dS_Q = \int_S \frac{\partial u(Q)}{\partial \mathbf{n}_Q} \ln |P - Q| dS_Q,$$

$$\lim_{A \rightarrow P} \int_S u(Q) \frac{\partial}{\partial \mathbf{n}_Q} \ln |A - Q| dS_Q = -\pi u(P) + \int_S u(Q) \frac{\partial}{\partial \mathbf{n}_Q} \ln |P - Q| dS_Q$$

Note that calculation of these limits involves calculation of the integrals in the **principal value** sense, otherwise divergent.

- We thus obtain

$$u(P) = \frac{1}{2\pi} \oint_S \left[\frac{\partial u(Q)}{\partial \mathbf{n}_Q} \ln |P - Q| - u(Q) \frac{\partial}{\partial \mathbf{n}_Q} \ln |P - Q| \right] dS_Q, \quad P \in S$$

which is a **boundary integral equation** connecting the **Dirichlet** and **Neumann** boundary data for a Laplace equation

BOUNDARY ELEMENT METHOD V

- Solution of the **interior Neumann** problem — given $\frac{\partial u}{\partial \mathbf{n}}|_S = f$, we can determine $u|_S$ from the relation

$$u(P) + \frac{1}{\pi} \oint_S u(Q) \frac{\partial}{\partial \mathbf{n}_Q} \ln |P - Q| dS_Q = \frac{1}{\pi} \oint_S \frac{\partial u(Q)}{\partial \mathbf{n}_Q} \ln |P - Q| dS_Q, \quad P \in S$$

- The **Boundary Element Method** relies on solving approximately the above equation to obtain $u|_S$ which is then used to calculate the solution of the Laplace equation in D
- The above is a **Fredholm Integral Equation of the Second Kind** which is **singular** in the sense that it admits solutions determined up to a constant (note that this fact reflects similar nonuniqueness of solutions of the interior Neumann problem)

BOUNDARY ELEMENT METHOD VI

- Solution of the **interior Dirichlet** problem — given $u|_S = f$, we can determine $\frac{\partial u}{\partial \mathbf{n}}|_S$ from the relation

$$\frac{1}{\pi} \oint_S \frac{\partial u(Q)}{\partial \mathbf{n}_Q} \ln |P - Q| dS_Q = g(P), \quad P \in S,$$

where

$$g(P) = f(P) + \frac{1}{\pi} \int_S u(Q) \frac{\partial}{\partial \mathbf{n}_Q} \ln |P - Q| dS_Q$$

- The **Boundary Element Method** relies on solving approximately the above equation to obtain $\frac{\partial u}{\partial \mathbf{n}}|_S$ which is then used to calculate the solution of the Laplace equation in D
- The above is a **Fredholm Integral Equation of the First Kind** which is known to be often ill-posed and therefore hard to solve numerically

BOUNDARY ELEMENT METHOD VII

- When solving the **interior Dirichlet** problem, the ill-posedness of the boundary integral equation can be circumvented representing the solution $u(A)$, $A \in D$ in terms of the **double layer potential** $\rho(Q)$, $Q \in S$ as

$$u(A) = \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \ln |P - Q| dS_Q$$

- The boundary distribution of the double layer potential can now be obtained from

$$-\pi\rho(P) + \oint_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \ln |P - Q| dS_Q = u(P), \quad P \in S$$

which is a **Fredholm Equation of the Second Kind** and is therefore well-posed.

- This approach is often referred to as **indirect**

BOUNDARY ELEMENT METHOD VIII

- Numerical methods (e.g., FEM) for approximate solution of the boundary integral equations are in general well-developed and well-understood (better for Fredholm equations of the second type)
- The Boundary Element Method offers the advantage of replacing a given problem with another one in a spatial dimension reduced by one (e.g., 1D boundary integral equations for 2D boundary value problem)
- The disadvantage, however, is that the resulting matrices are usually full
- The Boundary Element Method is particularly useful in solution of problems involving irregular domains which are hard to mesh
- In 2D elegant equivalent formulations can be derived using complex variables and complex-valued functions.

THE END —THANK YOU!!!