Part V

Boundary Element Method

Boundary Element Method I

- Boundary Element Method — alternative approach to solution of boundary value problems
- Motivation — consider the following initial value problem

$$\frac{du(t)}{dt} = f(u(t),t), \quad u(0) = u_0$$

This problem can be alternatively restated as

$$u(t) = u_0 + \int_0^t f(u(\tau),\tau)\, d\tau$$

- When solving the problem numerically, either of the forms can be taken as the point of departure
- The Boundary Element Method relies on an integral representation of the solution of an elliptic problem

Boundary Element Method II

- Let D be a bounded, open domain in $\mathbb{R}^2$ with a boundary S. At any point $P \in S$ the inner unit normal vector is denoted by $\mathbf{n}_P$; to fix attention, we will consider the following two boundary–value problems:
  - The interior Dirichlet problem — given $f \in C(S)$, find $u \in C(\overline{D}) \bigcap C^2(D)$ such that

$$\begin{cases} \Delta u(P) = 0, & P \in D \\ u(P) = f(P), & P \in S \end{cases}$$

  - The interior Neumann problem — given $f \in C(S)$, find $u \in C(\overline{D}) \bigcap C^2(D)$ such that

$$\begin{cases} \Delta u(P) = 0, & P \in D \\ \dfrac{\partial u(P)}{\partial \mathbf{n}_P} = f(P), & P \in S \end{cases}$$

- Generalizations to other boundary conditions, 3D domains and more complicated linear equations are possible.

Boundary Element Method III

- Given $u, w \in C^2(\overline{D})$, consider Green's identity (obtained from the divergence theorem

$$\int_D (u\Delta w - w\Delta u)\, d\Omega = \oint_S \left(w\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial w}{\partial \mathbf{n}} \right) dS$$

- Assume that u is the solution of the Laplace equation and $w = \frac{1}{2\pi} \ln|A-Q|$, i.e., the fundamental solution of the Laplace equation in an unbounded domain ($|A-Q|$ denotes the Euclidean norm of the vector $A-Q$); the above relation can be converted into

$$u(A) = \frac{1}{2\pi} \oint_S \left[\frac{\partial u(Q)}{\partial \mathbf{n}_Q} \ln|A-Q| - u(Q)\frac{\partial}{\partial \mathbf{n}_Q} \ln|A-Q| \right] dS_Q, \quad A \in D$$

- Thus, the solution of the Laplace equation can be represented as an integral involving both the Dirichlet and Neumann boundary conditions; note that only one of them is available.

Boundary Element Method IV

- Thus, if
  - $u(Q)$ is given, need to determine $\frac{\partial u(Q)}{\partial \mathbf{n}_Q}$,
  - $\frac{\partial u(Q)}{\partial \mathbf{n}_Q}$ is given, need to determine $u(Q)$
- When $A \in D$ approaches $P \in S$ the following limits can be calculated in the integral equation

$$\lim_{A\to P}\int_S \frac{\partial u(Q)}{\partial \mathbf{n}_Q}\ln|A-Q|\,dS_Q = \int_S \frac{\partial u(Q)}{\partial \mathbf{n}_Q}\ln|P-Q|\,dS_Q,$$

$$\lim_{A\to P}\int_S u(Q)\frac{\partial}{\partial \mathbf{n}_Q}\ln|A-Q|\,dS_Q = -\pi u(P) + \int_S u(Q)\frac{\partial}{\partial \mathbf{n}_Q}\ln|P-Q|\,dS_Q$$

  Note that calculation of these limits involves calculation of the integrals in the principal value sense, otherwise divergent.

- We thus obtain

$$u(P) = \frac{1}{2\pi}\oint_S\left[\frac{\partial u(Q)}{\partial \mathbf{n}_Q}\ln|P-Q| - u(Q)\frac{\partial}{\partial \mathbf{n}_Q}\ln|P-Q|\right]dS_Q, \quad P\in S$$

  which is a boundary integral equation connecting the Dirichlet and Neumann boundary data for a Laplace equation

Boundary Element Method V

- Solution of the interior Neumann problem — given $\frac{\partial u}{\partial \mathbf{n}}\big|_S = f$, we can determine $u|_S$ from the relation

$$u(P) + \frac{1}{\pi}\oint_S u(Q)\frac{\partial}{\partial \mathbf{n}_Q}\ln|P-Q|dS_Q = \frac{1}{\pi}\oint_S \frac{\partial u(Q)}{\partial \mathbf{n}_Q}\ln|P-Q|dS_Q, \quad P\in S$$

- The Boundary Element Method relies on solving approximately the above equation to obtain $u|_S$ which is then used to calculate the solution of the Laplace equation in D
- The above is a Fredholm Integral Equation of the Second Kind which is singular in the sense that it admits solutions determined up to a constant (note that this fact reflects similar nonuniqueness of solutions of the interior Neumann problem)

Boundary Element Method VI

- Solution of the interior Dirichlet problem — given $u|_S = f$, we can determine $\frac{\partial u}{\partial \mathbf{n}}\big|_S$ from the relation

$$\frac{1}{\pi}\oint_S \frac{\partial u(Q)}{\partial \mathbf{n}_Q}\ln|P-Q|\,dS_Q = g(P), \quad P\in S,$$

  where

$$g(P) = f(P) + \frac{1}{\pi}\int_S u(Q)\frac{\partial}{\partial \mathbf{n}_Q}\ln|P-Q|\,dS_Q$$

- The Boundary Element Method relies on solving approximately the above equation to obtain $\frac{\partial u}{\partial \mathbf{n}}\big|_S$ which is then used to calculate the solution of the Laplace equation in D
- The above is a Fredholm Integral Equation of the First Kind which is known to be often ill–posed and therefore hard to solve numerically

Boundary Element Method VII

- When solving the interior Dirichlet problem, the ill–posedness of the boundary integral equation can be circumvented representing the solution $u(A)$, $A\in D$ in terms of the double layer potential $\rho(Q)$, $Q\in S$ as

$$u(A) = \int_S \rho(Q)\frac{\partial}{\partial \mathbf{n}_Q}\ln|P-Q|\,dS_Q$$

- The boundary distribution of the double layer potential can now be obtained from

$$-\pi\rho(P) + \oint_S \rho(Q)\frac{\partial}{\partial \mathbf{n}_Q}\ln|P-Q|dS_Q = u(P), \quad P\in S$$

  which is a Fredholm Equation of the Second Kind and is therefore well–posed.

- This approach is often referred to as indirect

Boundary Element Method VIII

- Numerical methods (e.g., FEM) for approximate solution of the boundary integral equations are in general well–developed and well–understood (better for Fredholm equations of the second type)
- The Boundary Element Method offers the advantage of replacing a given problem with another one in a spatial dimension reduced by one (e.g., 1D boundary integral equations for 2D boundary value problem)
- The disadvantage, however, is that the resulting matrices are usually full
- The Boundary Element Method is particularly useful in solution of problems involving irregular domains which are hard to mesh
- In 2D elegant equivalent formulations can be derived using complex variables and complex–valued functions.

The End —Thank You!!!