

PART IV

Finite Element Method

FINITE ELEMENT METHOD I

- Computational properties of the method are largely determined by the properties of the algebraic system matrix A , in particular
 - For general sets of basis functions, the system matrix A will be full and may have prohibitively large condition number
 - Choosing the basis functions with small supports and without significant overlaps results in **sparse** system matrices which are easier to assemble and cheaper to solve (usually $O(N^2)$, instead of $O(N^3)$, operations)
- **The Finite Element Method** is a combination of:
 - The Galerkin approach, and
 - piecewise images of polynomials as the basis function
- Accuracy can be refined by:
 - refining the mesh (**h -refinement**)
 - increasing the order of the interpolating polynomials (**p -refinement**)
 - doing both at the same time (**hp -refinement**)

FINITE ELEMENT METHOD II

SIMPLE 1D EXAMPLE

- Consider the following Boundary Value Problem

$$\begin{cases} -u'' + u = f, & \text{in } \Omega, \\ u(0) = 0, \quad u'(1) = b, \end{cases}$$

where $f \in L_2(0, 1)$ and $b \in \mathbb{R}$

- Let

$$V = H_{(0)}^1(0, 1) = \{v \in H^1(0, 1), \mid v(0) = 0\}$$

The weak formulation is

$$u \in V, \quad \int_0^1 (u'v' + uv) dx = \int_0^1 fv dx + bv(1), \quad \forall v \in V$$

- Existence of an unique solution is guaranteed by the Lax–Milgram Lemma

FINITE ELEMENT METHOD III

SIMPLE 1D EXAMPLE

- Partition the domain $I = [0, 1]$ into N parts as $0 = x_0 < x_1 < \dots < x_N = 1$
 - The points x_i , $0 \leq i \leq N$ are called **nodes**,
 - The subintervals $I_i = [x_{i-1}, x_i]$, $1 \leq i \leq N$ are called **elements**

Denote $h_i = x_i - x_{i-1}$ and the **mesh parameter** $h = \max_{1 \leq i \leq N} h_i$.

- Approximate solution will be sought in the space $V_h = \{v_h \in V \mid v_h|_{I_i} \in P_1(I_i), 1 \leq i \leq N\}$; note that, given the properties of the Sobolev space $H^1(I)$, we also have $v_h \in C(\bar{I})$
- For the basis functions we choose

$$\phi_i(x) = \begin{cases} (x - x_{i-1})/h_i, & x_{i-1} \leq x \leq x_i, \\ (x_{i+1} - x)/h_{i+1}, & x_i \leq x \leq x_{i+1}, \\ 0, & \text{otherwise} \end{cases} \quad \text{for } i = 1, \dots, N-1$$

$$\phi_N(x) = \begin{cases} (x - x_{N-1})/h_N, & x_{N-1} \leq x \leq x_N, \\ 0, & \text{otherwise} \end{cases} \quad \text{for } i = N$$

FINITE ELEMENT METHOD IV

SIMPLE 1D EXAMPLE

- Note that thus defined basis functions are linearly independent and we have $V_h = \text{span}\{\phi_i, 1 \leq i \leq N\}$; their weak derivatives exist and are defined almost everywhere (equal to constants)

- **The Finite Element Method** thus becomes:

$$u_h \in V_h, \quad \int_0^1 (u_h' v_h' + u_h v_h) dx = \int_0^1 f v_h dx + b v_h(1), \quad \forall v_h \in V_h$$

which, using the representation $u_h = \sum_{i=1}^N u_i \phi_i$, can be transformed to the linear system

$$\sum_{j=1}^N u_j \int_0^1 (\phi_i' \phi_j' + \phi_i \phi_j) dx = \int_0^1 f \phi_i dx + b \phi_i(1), \quad 1 \leq i \leq N$$

- This system can be rewritten as $\mathbf{A}\mathbf{u} = \mathbf{b}$, where
 - $\mathbf{u} = (u_1, \dots, u_N)^T$ is the vector of unknown coefficients,
 - $\mathbf{b} = \left(\int_0^1 f \phi_1 dx, \dots, \int_0^1 f \phi_{N-1} dx, \int_0^1 f \phi_N dx + b \right)^T$ is the **load vector**

FINITE ELEMENT METHOD V

SIMPLE 1D EXAMPLE

- Entries of the **stiffness matrix** can be calculated as $A_{ij} = \int_0^1 (\phi'_i \phi'_j + \phi_i \phi_j) dx$;
Using the formulae

$$\int_0^1 \phi'_i \phi'_{i-1} dx = -\frac{1}{h}, \quad 2 \leq i \leq N, \quad \int_0^1 (\phi'_i)^2 dx = \frac{2}{h}, \quad 1 \leq i \leq N-1,$$

$$\int_0^1 \phi_i \phi_{i-1} dx = \frac{h}{6}, \quad 2 \leq i \leq N, \quad \int_0^1 (\phi_i)^2 dx = \frac{2h}{3}, \quad 1 \leq i \leq N-1,$$

$$\int_0^1 (\phi'_N)^2 dx = \frac{1}{h}, \quad \int_0^1 (\phi_N)^2 dx = \frac{h}{3},$$

we obtain for the stiffness matrix

$$A = \begin{bmatrix} \left(\frac{2h}{3} + \frac{2}{h}\right) & \left(\frac{h}{6} + \frac{1}{h}\right) & & & \\ \left(\frac{h}{6} + \frac{1}{h}\right) & \left(\frac{2h}{3} + \frac{2}{h}\right) & \left(\frac{h}{6} + \frac{1}{h}\right) & & \\ & \ddots & \ddots & \ddots & \\ & & \left(\frac{h}{6} + \frac{1}{h}\right) & \left(\frac{2h}{3} + \frac{2}{h}\right) & \left(\frac{h}{6} + \frac{1}{h}\right) \\ & & & \left(\frac{h}{6} + \frac{1}{h}\right) & \left(\frac{h}{3} + \frac{1}{h}\right) \end{bmatrix}$$

FINITE ELEMENT METHOD VI HIGHER-ORDER ELEMENTS

- Consider our original Boundary Value Problem; we will now use **piecewise quadratic functions** with the finite element space

$$V_h = \{v_h \in C(\bar{I}) \mid v_h|_{I_i} \text{ is quadratic, } v_h(0) = 0\}$$

- Denoting the **mid-points** of the subintervals as $x_{i-1/2} = \frac{x_{i-1} + x_i}{2}$, $1 \leq i \leq N$, we have the following sets of basis functions:

- associated with the nodes x_i , $1 \leq i \leq N - 1$

$$\phi_i(x) = \begin{cases} 2(x - x_{i-1})(x - x_{i-1/2})/h_i^2, & x \in [x_{i-1}, x_i], \\ 2(x_{i+1} - x)(x_{i+1/2} - x)/h_{i+1}^2, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise} \end{cases}$$

- associated with the nodes x_N

$$\phi_N(x) = \begin{cases} 2(x - x_{N-1})(x - x_{N-1/2})/h_N^2, & x \in [x_{N-1}, x_N], \\ 0, & \text{otherwise} \end{cases}$$

FINITE ELEMENT METHOD VII HIGHER-ORDER ELEMENTS

- Cont'd
 - associated with the $x_{i-1/2}$, $1 \leq i \leq N$

$$\psi_{i-1/2}(x) = \begin{cases} 4(x_i - x)(x - x_{i-1})/h_i^2, & x \in [x_{i-1}, x_i], \\ 0, & \text{otherwise} \end{cases}$$

- Thus, the finite element space can be represented as $V_h = \text{span}\{\phi_i, \psi_{i-1/2}, \mid 1 \leq i \leq N\}$ and we can write

$$u_h = \sum_{j=1}^N u_j \phi_j + \sum_{j=1}^N u_{j-1/2} \psi_{i-1/2}$$

- The Finite Element System is therefore

$$\begin{cases} a(u_h, \phi_i) = l(\phi_i), & 1 \leq i \leq N, \\ a(u_h, \psi_{i-1/2}) = l(\psi_{i-1/2}), & 1 \leq i \leq N, \end{cases} \text{ or } \begin{cases} M_{11} \mathbf{u} + M_{12} \hat{\mathbf{u}} = \mathbf{b}_1, \\ M_{21} \mathbf{u} + D_{22} \hat{\mathbf{u}} = \mathbf{b}_2, \end{cases}$$

where $\mathbf{u} = [u_1, \dots, u_N]^T$ and $\hat{\mathbf{u}} = [u_{1/2}, \dots, u_{N-1/2}]^T$

FINITE ELEMENT METHOD VIII HIGHER-ORDER ELEMENTS

- Note that
 - $M_{11} = [a(\phi_j, \phi_i)]_{N \times N}$ is a **tridiagonal** matrix,
 - $M_{12} = [a(\psi_{j-1/2}, \phi_i)]_{N \times N}$ is a **twodiagonal** matrix and $M_{21} = M_{12}^T$
 - $D_{22} = [a(\psi_{j-1/2}, \psi_{i-1/2})]_{N \times N}$ is a positive-definite **diagonal** matrix
- Thus, we can easily re-express $\hat{\mathbf{u}}$ as $\hat{\mathbf{u}} = D_{22}^{-1}(\mathbf{b}_2 - M_{21}\mathbf{u})$ leading to

$$M\mathbf{u} = \mathbf{b},$$

where

- $M = M_{11} - M_{12}D_{22}^{-1}M_{21}$ is a **tridiagonal** matrix
- $\mathbf{b} = \mathbf{b}_1 - M_{12}D_{22}^{-1}\mathbf{b}_2$
- The procedure of eliminating $\hat{\mathbf{u}}$ is known as **condensation** ; note that the size and structure of the resulting algebraic system remains the same!

FINITE ELEMENT METHOD IX NON-CONFORMING METHOD

- Consider the following fourth-order boundary value problem

$$\begin{cases} u^{(IV)} + u = f, & \text{in } \Omega, \\ u(0) = u'(0) = 0, & u'(1) = u''(1) = 0, \end{cases}$$

Assuming $V = H_0^2(0, 1)$, the weak formulation is

$$u \in V, \quad \int_0^1 u'' v'' dx = \int_0^1 f v dx, \quad \forall v \in V$$

- In the **conforming** case $V_h \subseteq V$, hence the V_h must be (at least) C^1 continuous; at every interior node we thus have two continuity conditions (i.e., for the function and its derivative); therefore, the order of the interpolating polynomial must be $p \geq 3$

FINITE ELEMENT METHOD X NON-CONFORMING METHOD

- The choice $p = 3$ offers the minimum number of required adjustable coefficients (4); in this case we have

$$V_h = \{v_h \in C^1(\bar{I}) \mid v_h|_{I_i} \in P_3(I_i), 1 \leq i \leq N, v_h(x) = v_h'(x) = 0 \text{ at } x = 0, 1\}$$

with the basis functions

$$\phi_i(x_j) = \delta_{ij}, \phi_i'(x_j) = 0,$$

$$\psi_i(x_j) = 0, \psi_i'(x_j) = \delta_{ij},$$

- In order to avoid the computational complexity of such **conforming** elements, one may use **non-conforming** elements, e.g., require C global continuity instead of C^1 , i.e.,

$$V_h = \{v_h \in C(\bar{I}) \mid v_h|_{I_i} \in P_2(I_i), 1 \leq i \leq N, v_h(x) = v_h'(x) = 0 \text{ at } x = 0, 1\}$$

- Note that in the **non-conforming** case we have $V_h \not\subseteq V$; nevertheless, in certain cases convergence of such approximations can still be assured

FINITE ELEMENT METHOD XI REFERENCE ELEMENT TECHNIQUE

- Consider again a **conforming** approach (with $p = 3$) to the solution of the fourth-order boundary value problem state earlier
- Introduce the **reference element** $I_0 = [0, 1]$ and a bijective mapping between I_0 and I_i , $1 \leq i \leq N$ defined as

$$F_i : I_0 \longrightarrow I_i, F_i(\xi) = x_{i-1} + h_i \xi$$

- Over the reference element I_0 we construct the cubic **shape functions** :

$$\Phi_0(\xi) = (1 + 2\xi)(1 - \xi)^2, \quad \Phi_1(\xi) = (3 - 2\xi)\xi^2,$$

$$\Psi_0(\xi) = \xi(1 - \xi)^2, \quad \Psi_1(\xi) = -(1 - \xi)\xi^2$$

Note that they satisfy by construction the interpolation conditions

$$\Phi_0(0) = 1, \quad \Phi_0(1) = 0, \quad \Phi'_0(0) = 0, \quad \Phi'_0(1) = 0,$$

$$\Phi_1(0) = 0, \quad \Phi_1(1) = 1, \quad \Phi'_1(0) = 0, \quad \Phi'_1(1) = 0,$$

$$\Psi_0(0) = 0, \quad \Psi_0(1) = 0, \quad \Psi'_0(0) = 1, \quad \Psi'_0(1) = 0,$$

$$\Psi_1(0) = 0, \quad \Psi_1(1) = 0, \quad \Psi'_1(0) = 0, \quad \Psi'_1(1) = 1,$$

FINITE ELEMENT METHOD XII REFERENCE ELEMENT TECHNIQUE

- Based on the shape functions defined on the reference element I_0 , we can easily construct the basis functions with the aid of the mappings $\{F_i\}_{i=1}^N$:

$$\phi_i(x) = \begin{cases} \Phi_1(F_i^{-1}(x)), & x \in I_i, \\ \Phi_0(F_{i+1}^{-1}(x)), & x \in I_{i+1}, \\ 0, & \text{otherwise} \end{cases}, \quad \psi_i(x) = \begin{cases} h_i \Psi_1(F_i^{-1}(x)), & x \in I_i, \\ h_{i+1} \Psi_0(F_{i+1}^{-1}(x)), & x \in I_{i+1}, \\ 0, & \text{otherwise} \end{cases}$$

- Computation of the entries of the stiffness matrix and the load vector is now performed in the reference space, e.g.,

$$a_{i-1,i} = \int_{I_i} (\phi_{i-1}'')(\phi_i'') dx = \int_{I_0} (\Phi_0)'' h_i^{-2} (\Phi_1)'' h_i^{-2} h_i d\xi$$

- The use of the [reference element technique](#) is essential both for deriving error estimates and for efficient implementation, especially in higher-dimensional cases

FINITE ELEMENT METHOD XIII

ESSENTIAL STEPS

- Essential steps need for solution of a boundary value problem with a **Finite Element Method**
 1. **weak formulation** of the boundary value problem
 2. **partition** (“triangulation”) of the solution domain into subdomains (“elements”)
 3. definition of a **finite element space** associated with this partition,
 4. construction of **basis functions** spanning the finite element space
 5. **assembly and solution** of the finite element system

FINITE ELEMENT METHOD XIV TRIANGULATION

- **Triangulation** is a partition $\mathcal{T}_h = \{K\}$ of the domain $\bar{\Omega}$ into a finite number of subsets K , called **elements**, with the following properties:
 - $\bigcup_{K \in \mathcal{T}_h} K = \bar{\Omega}$,
 - each K is closed with a nonempty interior \dot{K} and a Lipschitz continuous boundary
 - for distinct $K_1, K_2 \in \mathcal{T}_h$, $\dot{K}_1 \cap \dot{K}_2 = \emptyset$
 - for distinct $K_1, K_2 \in \mathcal{T}_h$, $K_1 \cap K_2$ is either empty, or a common vertex, or a common side of K_1 and K_2 (the **regularity condition**)
- We will now focus mostly on the 2D case (generalization to 3D is straight-forward, but may get technically complicated)
- We will assume that the domain Ω is a **polygon**, so it can be partitioned into straight-sided **triangles** and **quadrilaterals** (otherwise, curvilinear elements need to be used)
- 3D domains can be partitioned into tetrahedral, hexahedral, pentahedral, etc., elements

FINITE ELEMENT METHOD XV

TRIANGULATION

- The **finite elements** will be collectively denoted K , whereas the **reference element** will be denoted \hat{K} , so that we will have the bijective mapping $K = F_K(\hat{K})$; the function F_K is
 - **linear** when \hat{K} is a triangle (e.g., an equilateral one),
 - **bilinear** when \hat{K} is a quadrilateral

- For an arbitrary K we denote:

- $h_K = \text{diam}(K) = \max\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{x}, \mathbf{y} \in K\}$,
- ρ_K — the diameter of the largest circle (sphere) inscribed in K

Analogous quantities \hat{h} and $\hat{\rho}$ will be defined can be defined for the reference element \hat{K}

- The quantity h_K described the size of K
- The ratio $\frac{h_K}{\rho_K}$ measure the “flatness” of the element K

FINITE ELEMENT METHOD XVI

TRIANGULATION — UNSTRUCTURED GRIDS

- **Structured grid** , or **mesh** , is a grid in which every node, apart from the boundary nodes, has the same set of neighbors, i.e., the topology of the grid is the same in all parts of the domain
- The advantage of the Finite Element Method is that it does not need structured grids which are hard to generate in complex geometries
- An **unstructured grid** is simply a “cloud” of points which are appropriately numbered and connected; the resulting “finite elements” are often referred to as **simplices**
- In many case **grid generation** is as important and difficult as solution of the problem itself; hence grid generation is an autonomous area of research

FINITE ELEMENT METHOD XVII

DELAUNAY TRIANGULATION

- **Delaunay** triangulation is a viable possibility for unstructured grid generation, since it ensures that the resulting finite element are characterized by a moderate aspect ratio; this is in turn important from the point of view of interpolation errors (**Empty Circumcircle Property**)
- Definition — A triangulation \mathcal{T} is said to be **Delaunay** if, for every element $K \in \mathcal{T}$, the interior of the circumscribed sphere does not contain any vertex of the triangulation
- Properties of Delaunay triangulation:
 - All the simplices of a Delaunay triangulation contain the center of the circumscribed sphere
 - The empty circumcircle property is satisfied of every two simplices having a face in common
 - In 2D all the angles of a Delaunay triangulation are acute

FINITE ELEMENT METHOD XVIII

DELAUNAY TRIANGULATION

- Properties of Delaunay triangulation (cont'd):
 - Among all possible triangulations, the Delaunay triangulation \mathcal{T} maximizes the smallest interior angles of the simplices (denoted $\alpha(\mathcal{T})$) and minimizes the largest radius of the circumcircle (denoted $r(\mathcal{T})$)
 - The Delaunay triangulation is closely related to **Voronoi diagrams** (a.k.a. **Dirichlet tessellation**) in which every point on a plane is assigned a convex polygon such that this point is closer to the center of the polygon than any other point; Delaunay triangulations and Voronoi diagrams are dual in the graph-theoretic sense
- Delaunay triangulations can be generated in different ways, e.g., using the **Bowyer–Watson** algorithm

FINITE ELEMENT METHOD XIX

POLYNOMIAL SPACES ON REF'CE ELEMENTS

- With the aid of the mapping F_K , function spaces can be constructed on general elements from those constructed on the reference elements; thus we introduce a **polynomial space** \hat{X} on \hat{K}
- As a 2D example, consider an equilateral triangle \hat{K} with the vertices $\hat{A}_1(-1, 0)$, $\hat{A}_2(1, 0)$ and $\hat{A}_3(0, \sqrt{3})$; we can now introduce the functions (referred to as **barycentric coordinates** associated with the triangle \hat{K})

$$\hat{\lambda}_1(\hat{\mathbf{x}}) = \frac{1}{2} \left(1 - \hat{x} - \frac{\hat{y}}{\sqrt{3}} \right), \quad \hat{\lambda}_2(\hat{\mathbf{x}}) = \frac{1}{2} \left(1 + \hat{x} - \frac{\hat{y}}{\sqrt{3}} \right), \quad \hat{\lambda}_3(\hat{\mathbf{x}}) = \frac{\hat{y}}{\sqrt{3}},$$

Note that these linear functions satisfy $\hat{\lambda}_i(\hat{A}_j) = \delta_{ij}$

- Barycentric coordinates are convenient for representation of polynomials, e.g., any function $\hat{v} \in P_1(\hat{K})$ is determined by three parameters ($\hat{v} = \alpha_1 \hat{x}_1 + \alpha_2 \hat{x}_2 + \alpha_3$) and can be represented as

$$\hat{v}(\hat{\mathbf{x}}) = \sum_{i=1}^3 \hat{v}(\hat{A}_i) \hat{\lambda}_i(\hat{\mathbf{x}})$$

FINITE ELEMENT METHOD XX

POLYNOMIAL SPACES ON REF'CE ELEMENTS

- A quadratic function $\hat{v} \in P_2(\hat{K})$ is determined by six parameters, i.e., $\hat{v} = \beta_1 \hat{x}_1^2 + \beta_2 \hat{x}_1 \hat{x}_2 + \beta_3 \hat{x}_2^2 + \beta_4 \hat{x}_1 + \beta_5 \hat{x}_2 + \beta_6$, therefore six interpolation conditions are required
- Introduce **side mid-points** as $\hat{A}_{ij} = \frac{1}{2}(\hat{A}_i + \hat{A}_j)$, $1 \leq i \leq j \leq 3$
- Any function $\hat{v} \in P_2(\hat{K})$ is uniquely determined by its values at the vertices $\{\hat{A}_i\}_{i=1}^3$ and the side mid-points $\{\hat{A}_{ij}\}_{1 \leq i \leq j \leq 3}$ and has a representation

$$\hat{v}(\hat{\mathbf{x}}) = \sum_{i=1}^3 \hat{v}(\hat{A}_i) \hat{\lambda}_i(\hat{\mathbf{x}}) (2\hat{\lambda}_i(\hat{\mathbf{x}}) - 1) + \sum_{1 \leq i \leq j \leq 3} 4\hat{v}(\hat{A}_{ij}) \hat{\lambda}_i(\hat{\mathbf{x}}) \hat{\lambda}_j(\hat{\mathbf{x}})$$

Note that:

- $\hat{\lambda}_k(\hat{\mathbf{x}}) (2\hat{\lambda}_k(\hat{\mathbf{x}}) - 1)$, $1 \leq k \leq 3$ is a quadratic function equal to 1 at \hat{A}_k and to 0 at all other vertices and side mid-points,
- $4\hat{\lambda}_i(\hat{\mathbf{x}}) \hat{\lambda}_j(\hat{\mathbf{x}})$, $1 \leq i \leq j \leq 3$ is a quadratic function equal to 1 at \hat{A}_{ij} and to 0 at all other vertices and side mid-points
- It is possible to use derivatives, or other quantities, to construct the interpolation conditions

FINITE ELEMENT METHOD XXI

AFFINE–EQUIVALENT FINITE ELEMENTS

- When $\bar{\Omega}$ is a polygonal domain partitioned into straight–sided triangles and quadrilaterals K , then every element K is an image of the reference element \hat{K} under an invertible **affine** mapping $F_K : \hat{K} \rightarrow K$ of the form

$$F_K(\hat{\mathbf{x}}) = \mathbf{T}_K \hat{\mathbf{x}} + \mathbf{b}_K,$$

where \mathbf{T}_K is an invertible 2×2 mapping and \mathbf{b}_K is a translation vector.

- For every element we define the function space X_K by the formula

$$X_K = \hat{X} \circ F_K^{-1} = \{v \mid v = \hat{v} \circ F_K^{-1}, \hat{v} \in \hat{X}\}$$

F_K being affine, the degree of the spaces X_K and \hat{X} is the same. Moreover, we have $v(\mathbf{x}) = \hat{v}(\hat{\mathbf{x}})$, $\forall \mathbf{x} \in K$, $\hat{\mathbf{x}} \in \hat{K}$, with $\mathbf{x} = F_K(\hat{\mathbf{x}})$

- Using the nodes on \hat{K} we can introduce the corresponding nodes on K as

$$\mathbf{x}_i^K = F_K(\hat{\mathbf{x}}_i), \quad i = 1, \dots, I$$

and the associated functions on K as

$$\phi_i^K = \hat{\phi}_i \circ F_K^{-1}, \quad i = 1, \dots, I$$

FINITE ELEMENT METHOD XXII

AFFINE-EQUIVALENT FINITE ELEMENTS

- Note that the functions $\{\phi_i^K\}_{i=1}^I$ have the property that

$$\phi_i^K(\mathbf{x}_j^K) = \delta_{ij},$$

hence they form a set of **local polynomials basis functions on K**

- Estimates of interpolation errors will depend on the properties of the affine transformation \mathbf{T}_K , for which we have the following result

$$\|\mathbf{T}_K\| \leq \frac{h_K}{\hat{\rho}}, \quad \|\mathbf{T}_K^{-1}\| \leq \frac{\hat{h}}{\rho_K}$$

FINITE ELEMENT METHOD XXIII

FINITE ELEMENT SPACES

- For second-order boundary value problems we need $V_h \subset H^1(\Omega)$; we may thus set $V_h = X_h$, where

$$X_h = \{v_h \in C(\bar{\Omega}) \mid v_h|_K \in X_K, \forall K \in \mathcal{T}_h\}$$

Function in X_h thus have to be **continuous** across element boundaries

- We can define the following interpolation operators:
 - On the reference element \hat{K} : $\hat{\Pi} : C(\hat{K}) \rightarrow \hat{X}$, $\hat{\Pi}\hat{v} = \sum_{i=1}^I \hat{v}(\hat{\mathbf{x}}_i)\hat{\phi}_i$
with the interpolation conditions $\hat{\Pi}\hat{v}(\hat{\mathbf{x}}_i) = \hat{v}(\hat{\mathbf{x}}_i)$, $i = 1, \dots, I$
 - On any actual element K : $\Pi_K : C(K) \rightarrow X_K$, $\Pi_K v = \sum_{i=1}^I v(\mathbf{x}_i^K)\phi_i^K$
with the interpolation conditions $\Pi_K v(\mathbf{x}_i^K) = v(\mathbf{x}_i^K)$, $i = 1, \dots, I$
- The two interpolation operators are related as $\hat{\Pi}\hat{v} = (\Pi_K v) \circ F_K^{-1} = \widehat{\Pi}_K v$, relation that is essential for error analysis.

FINITE ELEMENT METHOD XXIV

QUADRATURES

- When solving the approximate problem in the form

$$\text{find } u_h \in V_h, \quad a_h(u_h, v_h) = f_h(v_h) \quad \forall v_h \in V_h$$

we need to evaluate integrals

$$a_h(u_h, v_h) = \int_{\Omega} A_h(x, u_h, v_h) dx, \quad f_h(v_h) = \int_{\Omega} F_h(x, v_h) dx + \int_{\Gamma} G_h(x, v_h) dx,$$

where A_h , F_h , and G_h are operators. In most practical situations these integrals cannot be evaluated analytically and approximate approaches need to be used.

- **Quadrature** is a method to evaluate an integral approximately.
- **Gaussian Quadrature** seeks to obtain the best numerical estimate of an integral by picking optimal points x_i , $i = 1, \dots, N$ at which to evaluate the function $f(x)$.
- **The fundamental theorem of Gaussian quadrature** states that the optimal abscissas of the N -point Gaussian quadrature formulas are precisely the roots of the orthogonal polynomial for the same interval and weighting function

FINITE ELEMENT METHOD XXV

QUADRATURES

- Definition — Let K be a non-empty, Lipschitz, compact subset of \mathbb{R}^d . Let $l_q \geq 1$ be an integer. A quadrature on K with l_q points consists of
 - A set of l_q real numbers $\{\omega_1, \dots, \omega_{l_q}\}$ called **quadrature weights**
 - A set of l_q points $\{\xi_1, \dots, \xi_{l_q}\}$ in K called **Gauß points** or **quadrature nodes**

The largest integer k such that $\forall p \in P_k, \int_K p(x) dx = \sum_{l=1}^{l_q} \omega_l p(\xi_l)$ is called the **quadrature order** and is denoted by k_q

- As regards 1D bounded intervals, the most frequently used quadratures are based on **Legendre polynomials** which are defined on the interval $(0, 1)$ as $\mathcal{E}_k(t) = \frac{1}{k!} \frac{d^k}{dt^k} (t^2 - t)^k, k \geq 0$. Note that they are orthogonal on $(0, 1)$ with the weight $W = 1$.

FINITE ELEMENT METHOD XXVI QUADRATURES

- Theorem — Let $l_q \geq 1$, denote by ξ_1, \dots, ξ_{l_q} the l_q roots of the Legendre polynomial $\mathcal{E}_{l_q}(x)$ and set $\omega_l = \int_0^1 \prod_{\substack{j=1 \\ j \neq l}}^{l_q} \frac{t - \xi_j}{\xi_l - \xi_j} dt$. Then

$\{\xi_1, \dots, \xi_{l_q}, \omega_1, \dots, \omega_{l_q}\}$ is a quadrature of order $k_q = 2l_q - 1$ on $[0, 1]$

Proof — Let $\{\mathcal{L}_1, \dots, \mathcal{L}_{l_q}\}$ be the set of Lagrange polynomials associated with the Gauss points $\{\xi_1, \dots, \xi_{l_q}\}$. Then $\omega_l = \int_0^1 \mathcal{L}_l(t) dt$, $1 \leq l \leq l_q$

- when $p(x)$ is a polynomial of degree less than l_q , we integrate both sides of the identity $p(t) = \sum_{l=1}^{l_q} p(\xi_l) \mathcal{L}_l(t) dx$, $\forall t \in [0, 1]$ and deduce that the quadrature is exact for $p(x)$
- when the polynomial $p(x)$ has degree less than $2l_q$ we write it in the form $p(x) = q(x)\mathcal{E}_{l_q}(x) + r(x)$, where both $q(x)$ and $r(x)$ are polynomials of degree less than l_q ; owing to orthogonality of the Legendre polynomials, we conclude

$$\int_0^1 p(t) dt = \int_0^1 r(t) dt = \sum_{l=1}^{l_q} \omega_l r(\xi_l) = \sum_{l=1}^{l_q} \omega_l p(\xi_l),$$

since the points ξ_l are also roots of \mathcal{E}_{l_q}

FINITE ELEMENT METHOD XXVII

QUADRATURES

- The Gauß points and the weights are available in closed form only for low order quadratures; for higher order quadratures they have to be determined approximately, but are readily available in tables
- Generalization to 2D and 3D is straightforward; the Gauß points and weights are again available in tables
- When assembling FEM matrices, the following type need to be evaluated

$$\int_{\Omega} \phi(x) dx = \sum_{K \in \mathcal{T}_h} \int_K \phi(x) dx$$

Since the mapping $F_K : \hat{K} \rightarrow K$ is smooth, the change of variables $x = F_K(\hat{x})$ yields

$$\int_K \phi(x) dx = \int_{\hat{K}} \phi(F_K(\hat{x})) \det(J_K(\hat{x})) d\hat{x},$$

where $J_K(\hat{x}) = \frac{\partial F_K(\hat{x})}{\partial \hat{x}}$ is the **Jacobian** matrix of F_K at \hat{x} .

FINITE ELEMENT METHOD XXVIII QUADRATURES

- Thus, the integrals can be evaluated over the reference elements; using a quadrature with l_q Gauß points and weights on \hat{K} we obtain

$$\int_K \phi(x) dx \approx \sum_{l=1}^{l_q} \omega_l \det(J_K(\xi_l)) \phi(F_K(\xi_l)) = \sum_{l=1}^{l_q} \omega_{lK} \phi(\xi_{lK}),$$

where we set $\omega_{lK} = \omega_l \det(J_K(\xi_l))$ and $\xi_{lK} = F_K(\xi_l)$

- It can be shown that the quadrature $\{\xi_{1K}, \dots, \xi_{l_q K}, \omega_{1K}, \dots, \omega_{l_q K}\}$ enjoy similar properties regarding accuracy as the quadratures defined on K