Part IV

Finite Element Method

Finite Element Method I

- Computational properties of the method are largely determined by the properties of the algebraic system matrix A, in particular
  - For general sets of basis functions, the system matrix A will be full and may have prohibitively large condition number
  - Choosing the basis functions with small supports and without significant overlaps results in sparse system matrices which are easier to assemble and cheaper to solve (usually $\mathcal{O}(N^2)$, instead of $\mathcal{O}(N^3)$, operations)
- The Finite Element Method is a combination of:
  - The Galerkin approach, and
  - piecewise images of polynomials as the basis function
- Accuracy can be refined by:
  - refining the mesh (h–refinement)
  - increasing the order of the interpolating polynomials (p–refinement)
  - doing both at the same time (hp–refinement)

Finite Element Method II
Simple 1D Example

- Consider the following Boundary Value Problem

$$\begin{cases} -u'' + u = f, & \text{in } \Omega, \\ u(0) = 0, \ u'(1) = b, \end{cases}$$

  where $f \in L_2(0,1)$ and $b \in \mathbb{R}$

- Let

$$V = H^1_{(0}(0,1) = \{v \in H^1(0,1), \mid v(0) = 0\}$$

  The weak formulation is

$$u \in V, \quad \int_0^1 (u'v' + uv)\,dx = \int_0^1 fv\,dx + bv(1), \quad \forall v \in V$$

- Existence of an unique solution is guaranteed by the Lax–Milgram Lemma

Finite Element Method III
Simple 1D Example

- Partition the domain $I = [0,1]$ into N parts as $0 = x_0 < x_1 < \cdots < x_N = 1$
  - The points x_i, $0 \le i \le N$ are called nodes ,
  - The subintervals $I_i = [x_{i-1}, x_i]$, $1 \le i \le N$ are called elements

  Denote $h_i = x_i - x_{i-1}$ and the mesh parameter $h = \max_{1 \le i \le N} h_i$.
- Approximate solution will be sought in the space $V_h = \{v_h \in V \mid v_h|_{I_i} \in P_1(I_i), \ 1 \le i \le N\}$; note that, given the properties of the Sobolev space $H^1(I)$, we also have $v_h \in C(\bar{I})$
- For the basis functions we choose

$$\phi_i(x) = \begin{cases} (x - x_{i-1})/h_i, & x_{i-1} \le x \le x_i, \\ (x_{i+1} - x)/h_{i+1}, & x_i \le x \le x_{i+1}, \\ 0, & \text{otherwise} \end{cases} \quad \text{for } i = 1, \dots, N-1$$

$$\phi_N(x) = \begin{cases} (x - x_{N-1})/h_N, & x_{N-1} \le x \le x_N, \\ 0, & \text{otherwise} \end{cases} \quad \text{for } i = N$$

FINITE ELEMENT METHOD IV
SIMPLE 1D EXAMPLE

- Note that thus defined basis functions are linearly independent and we have $V_h = \text{span}\{\phi_i,\ 1 \le i \le N\}$; their weak derivatives exist and are defined almost everywhere (equal to constants)
- The Finite Element Method thus becomes:

$$u_h \in V_h, \quad \int_0^1 (u_h' v_h' + u_h v_h)\, dx = \int_0^1 f v_h\, dx + b v_h(1), \quad \forall v_h \in V_h$$

which, using the representation $u_h = \sum_{i=1}^N u_j \phi_j$, can be transformed to the linear system

$$\sum_{i=1}^N u_j \int_0^1 (\phi_i' \phi_j' + \phi_i \phi_j)\, dx = \int_0^1 f \phi_i\, dx + b \phi_i(1), \quad 1 \le i \le N$$

- This system can be rewritten as $A\mathbf{u} = \mathbf{b}$, where
  - $\mathbf{u} = (u_1, \dots, u_N)^T$ is the vector of unknown coefficients,
  - $\mathbf{b} = \left(\int_0^1 f\phi_i\, dx, \dots, \int_0^1 f\phi_{N-1}\, dx, \int_0^1 f\phi_N\, dx + b \right)^T$ is the load vector

FINITE ELEMENT METHOD V
SIMPLE 1D EXAMPLE

- Entries of the stiffness matrix can be calculated as $A_{ij} = \int_0^1 (\phi_i' \phi_j' + \phi_i \phi_j)\, dx$; Using the formulae

$$\int_0^1 \phi_i' \phi_{i-1}'\, dx = -\frac{1}{h}, \quad 2 \le i \le N, \qquad \int_0^1 (\phi_i')^2\, dx = \frac{2}{h}, \quad 1 \le i \le N-1,$$

$$\int_0^1 \phi_i \phi_{i-1}\, dx = \frac{h}{6}, \quad 2 \le i \le N, \qquad \int_0^1 (\phi_i)^2\, dx = \frac{2h}{3}, \quad 1 \le i \le N-1,$$

$$\int_0^1 (\phi_N')^2\, dx = \frac{1}{h}, \qquad \int_0^1 (\phi_N)^2\, dx = \frac{h}{3},$$

we obtain for the stiffness matrix

$$A = \begin{bmatrix} (\frac{2h}{3} + \frac{2}{h}) & (\frac{h}{6} + \frac{1}{h}) & & & \\ (\frac{h}{6} + \frac{1}{h}) & (\frac{2h}{3} + \frac{2}{h}) & (\frac{h}{6} + \frac{1}{h}) & & \\ \ddots & \ddots & \ddots & & \\ & (\frac{h}{6} + \frac{1}{h}) & (\frac{2h}{3} + \frac{2}{h}) & (\frac{h}{6} + \frac{1}{h}) \\ & & (\frac{h}{6} + \frac{1}{h}) & (\frac{h}{3} + \frac{1}{h}) \end{bmatrix}$$

FINITE ELEMENT METHOD VI
HIGHER–ORDER ELEMENTS

- Consider our original Boundary Value Problem; we will now use piecewise quadratic functions with the finite element space

$$V_h = \{v_h \in C(\bar{I}) \,|\, v_h|_{I_i} \text{ is quadratic},\ v_h(0) = 0\}$$

- Denoting the mid–points of the subintervals as $x_{i-1/2} = \frac{x_{i-1}+x_i}{2}$, $1 \le i \le N$, we have the following sets of basis functions:
  - associated with the nodes x_i, $1 \le i \le N-1$

$$\phi_i(x) = \begin{cases} 2(x - x_{i-1})(x - x_{i-1/2})/h_i^2, & x \in [x_{i-1}, x_i], \\ 2(x_{i+1} - x)(x_{i+1/2} - x)/h_{i+1}^2, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise} \end{cases}$$

  - associated with the nodes x_N

$$\phi_N(x) = \begin{cases} 2(x - x_{N-1})(x - x_{N-1/2})/h_N^2, & x \in [x_{N-1}, x_N], \\ 0, & \text{otherwise} \end{cases}$$

FINITE ELEMENT METHOD VII
HIGHER–ORDER ELEMENTS

- Cont'd
  - associated with the $x_{i-1/2}$, $1 \le i \le N$

$$\psi_{i-1/2}(x) = \begin{cases} 4(x_i - x)(x - x_{i-1})/h_i^2, & x \in [x_{i-1}, x_i], \\ 0, & \text{otherwise} \end{cases}$$

- Thus, the finite element space can be represented as $V_h = \text{span}\{\phi_i, \psi_{i-1/2}, \,|\, 1 \le i \le N\}$ and we can write

$$u_h = \sum_{j=1}^N u_j \phi_j + \sum_{j=1}^N u_{j-1/2} \psi_{i-1/2}$$

- The Finite Element System is therefore

$$\begin{cases} a(u_h, \phi_i) = l(\phi_i), & 1 \le i \le N, \\ a(u_h, \psi_{i-1/2}) = l(\psi_{i-1/2}), & 1 \le i \le N, \end{cases} \quad \text{or} \quad \begin{cases} M_{11}\mathbf{u} + M_{12}\hat{\mathbf{u}} = \mathbf{b}_1, \\ M_{21}\mathbf{u} + D_{22}\hat{\mathbf{u}} = \mathbf{b}_2, \end{cases}$$

where $\mathbf{u} = [u_1, \dots, u_N]^T$ and $\hat{\mathbf{u}} = [u_{1/2}, \dots, u_{N-1/2}]^T$

Finite Element Method VIII
Higher–Order Elements

- Note that
  - $M_{11} = [a(\phi_j, \phi_i)]_{N\times N}$ is a tridiagonal matrix,
  - $M_{12} = [a(\psi_{j-1/2}, \phi_i)]_{N\times N}$ is a twodiagonal matrix and $M_{21} = M_{12}^T$
  - $D_{22} = [a(\psi_{j-1/2}, \psi_{i-1/2})]_{N\times N}$ is a positive–definite diagonal matrix
- Thus, we can easily re–express $\hat{\mathbf{u}}$ as $\hat{\mathbf{u}} = D_{22}^{-1}(\mathbf{b}_2 - M21\mathbf{u})$ leading to

$$M\mathbf{u} = \mathbf{b},$$

  where
  - $M = M_{11} - M_{12}D_{22}^{-1}M_{21}$ is a tridiagonal matrix
  - $\mathbf{b} = \mathbf{b}_1 - M_{12}D_{22}^{-1}\mathbf{b}_2$
- The procedure of eliminating $\hat{\mathbf{u}}$ is known as condensation ; note that the size and structure of the resulting algebraic system remains the same!

Finite Element Method IX
Non–Conforming Method

- Consider the following fourth–order boundary value problem

$$\begin{cases} u^{(IV)} + u = f, & \text{in } \Omega, \\ u(0) = u'(0) = 0, & u'(1) = u'(1) = 0, \end{cases}$$

  Assuming $V = H_0^2(0,1)$, the weak formulation is

$$u \in V, \quad \int_0^1 u''v''\,dx = \int_0^1 fv\,dx, \quad \forall v \in V$$

- In the conforming case $V_h \subseteq V$, hence the V_h must be (at least) C^1 continuous; at every interior node we thus have two continuity conditions (i.e., for the function and its derivative); therefore, the order of the interpolating polynomial must be $p \geq 3$

Finite Element Method X
Non–Conforming Method

- The choice $p = 3$ offers the minimum number of required adjustable coefficients (4); in this case we have

$$V_h = \{v_h \in C^1(\bar{I})\,|\,v_h|_{I_i} \in P_3(I_i),\, 1 \leq i \leq N,\, v_h(x) = v_h'(x) = 0 \,\text{at}\, x = 0,1\}$$

  with the basis functions

$$\phi_i(x_j) = \delta_{ij}, \phi_i'(x_j) = 0,$$
$$\psi_i(x_j) = 0,\; \phi_i'(x_j) = \delta_{ij},$$

- In order to avoid the computational complexity of such conforming elements, one may use non–conforming elements, e.g., require C global continuity instead of C^1, i.e.,

$$V_h = \{v_h \in C(\bar{I})\,|\,v_h|_{I_i} \in P_2(I_i),\, 1 \leq i \leq N,\, v_h(x) = v_h'(x) = 0 \,\text{at}\, x = 0,1\}$$

- Note that in the non–conforming case we have $V_h \nsubseteq V$; nevertheless, in certain cases convergence of such approximations can still be assured

Finite Element Method XI
Reference Element Technique

- Consider again a conforming approach (with $p = 3$) to the solution of the fourth–order boundary value problem state earlier
- Introduce the reference element $I_0 = [0,1]$ and a bijective mapping between I_0 and I_i, $1 \leq i \leq N$ defined as

$$F_i : I_0 \longrightarrow I_i,\; F_i(\xi) = x_{i-1} + h_i\xi$$

- Over the reference element I_0 we construct the cubic shape functions :

$$\Phi_0(\xi) = (1+2\xi)(1-\xi)^2, \qquad \Phi_1(\xi) = (3-2\xi)\xi^2,$$
$$\Psi_0(\xi) = \xi(1-\xi)^2, \qquad \Psi_1(\xi) = -(1-\xi)\xi^2$$

  Note that they satisfy by construction the interpolation conditions

$\Phi_0(0) = 1,$	$\Phi_0(1) = 0,$	$\Phi_0'(0) = 0,$	$\Phi_0'(1) = 0,$
$\Phi_1(0) = 0,$	$\Phi_1(1) = 1,$	$\Phi_1'(0) = 0,$	$\Phi_1'(1) = 0,$
$\Psi_0(0) = 0,$	$\Psi_0(1) = 0,$	$\Psi_0'(0) = 1,$	$\Psi_0'(1) = 0,$
$\Psi_1(0) = 0,$	$\Psi_1(1) = 0,$	$\Psi_1'(0) = 0,$	$\Psi_1'(1) = 1,$

Finite Element Method XII
Reference Element Technique

- Based on the shape functions defined on the reference element I_0, we can easily construct the basis functions with the aid of the mappings $\{F_i\}_{i=1}^N$:

$$\phi_i(x) = \begin{cases} \Phi_1(F_i^{-1}(x)), & x \in I_i, \\ \Phi_0(F_{i+1}^{-1}(x)), & x \in I_{i+1}, \\ 0, & \text{otherwise} \end{cases}, \quad \psi_i(x) = \begin{cases} h_i\Psi_1(F_i^{-1}(x)), & x \in I_i, \\ h_{i+1}\Psi_0(F_{i+1}^{-1}(x)), & x \in I_{i+1}, \\ 0, & \text{otherwise} \end{cases}$$

- Computation of the entries of the stiffness matrix and the load vector is now performed in the reference space, e.g.,

$$a_{i-1,i} = \int_{I_i} (\phi''_{i-1})(\phi''_i)\,dx = \int_{I_0} (\Phi_0)'' h_i^{-2} (\Phi_1)'' h_i^{-2} h_i\,d\xi$$

- The use of the reference element technique is essential both for deriving error estimates and for efficient implementation, especially in higher–dimensional cases

Finite Element Method XIII
Essential Steps

- Essential steps need for solution of a boundary value problem with a Finite Element Method
  1. weak formulation of the boundary value problem
  2. partition ("triangulation") of the solution domain into subdomains ("elements")
  3. definition of a finite element space associated with this partition,
  4. construction of basis functions spanning the finite element space
  5. assembly and solution of the finite element system

Finite Element Method XIV
Triangulation

- Triangulation is a partition $\mathcal{T}_h = \{K\}$ of the domain $\overline{\Omega}$ into a finite number of subsets K, called elements, with the following properties:
  - $\bigcup_{K\in\mathcal{T}_h} K = \overline{\Omega}$,
  - each K is closed with a nonempty interior $\dot{K}$ and a Lipschitz continuous boundary
  - for distinct $K_1, K_2 \in \mathcal{T}_h$, $\dot{K}_1 \bigcap \dot{K}_2 = \emptyset$
  - for distinct $K_1, K_2 \in \mathcal{T}_h$, $K_1 \bigcap K_2$ is either empty, or a common vertex, or a common side of K_1 and K_2 (the regularity condition)
- We will now focus mostly on the 2D case (generalization to 3D is straight–forward, but may get technically complicated)
- We will assume that the domain Ω is a polygon , so it can be partitioned into straight–sided triangles and quadrilaterals (otherwise, curvilinear elements need to be used)
- 3D domains can be partitioned into tetrahedral, hexahedral, pentahedral, etc., elements

Finite Element Method XV
Triangulation

- The finite elements will be collectively denoted K, whereas the reference element will be denoted $\hat{K}$, so that we will have the bijective mapping $K = F_K(\hat{K})$; the function F_K is
  - linear when $\hat{K}$ is a triangle (e.g., an equilateral one),
  - bilinear when $\hat{K}$ is a quadrilateral
- For an arbitrary K we denote:
  - $h_K = \operatorname{diam}(K) = \max\{\|\mathbf{x}-\mathbf{y}\| \,|\, \mathbf{x},\mathbf{y} \in K\}$,
  - ρ_K — the diameter of the largest circle (sphere) inscribed in K

  Analogous quantities $\hat{h}$ and $\hat{\rho}$ will be defined can be defined for the reference element $\hat{K}$
- The quantity h_K described the size of K
- The ratio $\frac{h_K}{\rho_K}$ measure the "flatness" of the element K

Finite Element Method XVI
Triangulation — Unstructured Grids

- Structured grid , or mesh , is a grid in which every node, apart from the boundary nodes, has the same set of neighbors, i.e., the topology of the grid is the same in all parts of the domain
- The advantage of the Finite Element Method is that it does not need structured grids which are hard to generate in complex geometries
- An unstructured grid is simply a "cloud" of points which are appropriately numbered and connected; the resulting "finite elements" are often referred to as simplices
- In many case grid generation is as important and difficult as solution of the problem itself; hence grid generation is an autonomous area of research

Finite Element Method XVII
Delaunay Triangulation

- Delaunay triangulation is a viable possibility for unstructured grid generation, since it ensures that the resulting finite element are characterized by a moderate aspect ration; this is in turn important from the point of view of interpolation errors (Empty Circumcircle Property)
- Definition — A triangulation $\mathcal{T}$ is said to be Delaunay if, for every element $K \in \mathcal{T}$, the interior of the circumscribed sphere does not contain any vertex of the triangulation
- Properties of Delaunay triangulation:
  - All the simplices of a Delaunay triangulation contain the center of the circumscribed sphere
  - The empty circumcircle property is satisfied of every two simplices having a face in common
  - In 2D all the angles of a Delaunay triangulation are acute

Finite Element Method XVIII
Delaunay Triangulation

- Properties of Delaunay triangulation (cont'd):
  - Among all possible triangulations, the Delaunay triangulation $\mathcal{T}$ maximizes the smallest interior angles of the simplices (denoted $\alpha(\mathcal{T})$) and minimizes the largest radius of the circumcircle (denoted $r(\mathcal{T})$)
  - The Delaunay triangulation is closely related to Voronoi diagrams (a.k.a. Dirichlet tessellation) in which every point on a plane is assigned a convex polygon such that this point is closer to the center of the polygon than any other point; Delaunay triangulations and Voronoi diagrams are dual in the graph-theoretic sense
- Delaunay triangulations can be generated in different ways, e.g., using the Bowyer–Watson algorithm

Finite Element Method XIX
Polynomial Spaces on Ref'ce Elements

- With the aid of the mapping F_K, function spaces can be constructed on general elements from those constructed on the reference elements; thus we introduce a polynomial space $\hat{X}$ on $\hat{K}$
- As a 2D example, consider an equilateral triangle $\hat{K}$ with the vertices $\hat{A}_1(-1,0)$, $\hat{A}_2(1,0)$ and $\hat{A}_3(0,\sqrt{3})$; we can now introduce the functions (referred to as barycentric coordinates associated with the triangle $\hat{K}$)

$$\hat{\lambda}_1(\hat{\mathbf{x}}) = \frac{1}{2}\left(1 - \hat{x}_1 - \frac{\hat{x}_2}{\sqrt{3}}\right), \quad \hat{\lambda}_2(\hat{\mathbf{x}}) = \frac{1}{2}\left(1 + \hat{x}_1 - \frac{\hat{x}_2}{\sqrt{3}}\right), \quad \hat{\lambda}_3(\hat{\mathbf{x}}) = \frac{\hat{x}_2}{\sqrt{3}},$$

  Note that these linear functions satisfy $\hat{\lambda}_i(\hat{A}_j) = \delta_{ij}$
- Barycentric coordinates are convenient for representation of polynomials, e.g., any function $\hat{v} \in P_1(\hat{K})$ is determined by three parameters ($\hat{v} = \alpha_1 \hat{x}_1 + \alpha_2 \hat{x}_2 + \alpha_3$) and can be represented as

$$\hat{v}(\hat{\mathbf{x}}) = \sum_{i=1}^{3} \hat{v}(\hat{A}_i)\hat{\lambda}_i(\hat{\mathbf{x}})$$

Finite Element Method XX
Polynomial Spaces on Ref'ce Elements

- A quadratic function $\hat{v} \in P_2(\hat{K})$ is determined by six parameters, i.e., $\hat{v} = \beta_1 \hat{x}_1^2 + \beta_2 \hat{x}_1 \hat{x}_2 + \beta_3 \hat{x}_2^2 + \beta_4 \hat{x}_1 + \beta_5 \hat{x}_2 + \beta_6$, therefore six interpolation conditions are required
- Introduce side mid–points as $\hat{A}_{ij} = \frac{1}{2}(\hat{A}_i + \hat{A}_j)$, $1 \le i \le j \le 3$
- Any function $\hat{v} \in P_2(\hat{K})$ is uniquely determined by its values at the vertices $\{\hat{A}_i\}_{i=1}^3$ and the side mid–points $\{\hat{A}_{ij}\}_{1 \le i \le j \le 3}$ and has a representation

$$\hat{v}(\hat{\mathbf{x}}) = \sum_{i=1}^{3} \hat{v}(\hat{A}_i)\hat{\lambda}_i(\hat{\mathbf{x}})(2\hat{\lambda}_i(\hat{\mathbf{x}}) - 1) + \sum_{1 \le i \le j \le 3} 4\hat{v}(\hat{A}_{ij})\hat{\lambda}_i(\hat{\mathbf{x}})\hat{\lambda}_j(\hat{\mathbf{x}})$$

  Note that:
  - $\hat{\lambda}_k(\hat{\mathbf{x}})(2\hat{\lambda}_k(\hat{\mathbf{x}}) - 1)$, $1 \le k \le 3$ is a quadratic function equal to 1 at $\hat{A}_k$ and to 0 at all other vertices and side mid–points,
  - $4\hat{\lambda}_i(\hat{\mathbf{x}})\hat{\lambda}_j(\hat{\mathbf{x}})$, $1 \le i \le j \le 3$ is a quadratic function equal to 1 at $\hat{A}_{ij}$ and to 0 at all other vertices and side mid–points
- It is possible to use derivatives, or other quantities, to construct the interpolation conditions

Finite Element Method XXI
Affine–Equivalent Finite Elements

- When $\overline{\Omega}$ is a polygonal domain partitioned into straight–sided triangles and quadrilaterals K, then every elements K is an image of the reference element $\hat{K}$ under an invertible affine mapping $F_K : \hat{K} \to K$ of the form

$$F_K(\hat{\mathbf{x}}) = \mathbf{T}_K \hat{x} + \mathbf{b}_K,$$

  where $\mathbf{T}_K$ is an invertible 2×2 mapping and $\mathbf{b}_K$ is a translation vector.
- For every element we define the function space X_K by the formula

$$X_K = \hat{X} \circ F_K^{-1} = \{v \,|\, v = \hat{v} \circ F_K^{-1},\ \hat{v} \in \hat{X}\}$$

  F_K being affine, the degree of the spaces X_K and $\hat{X}$ is the same. Moreover, we have $v(\mathbf{x}) = \hat{v}(\hat{\mathbf{x}})$, $\forall \mathbf{x} \in K$, $\hat{\mathbf{x}} \in \hat{K}$, with $\mathbf{x} = F_K(\hat{\mathbf{x}})$
- Using the nodes on $\hat{K}$ we can introduce the corresponding nodes on K as

$$\mathbf{x}_i^K = F_K(\hat{\mathbf{x}}_i), \quad i = 1, \ldots, I$$

  and the associated functions on K as

$$\phi_i^K = \hat{\phi}_i \circ F_K^{-1}, \quad i = 1, \ldots, I$$

Finite Element Method XXII
Affine–Equivalent Finite Elements

- Note that the functions $\{\phi_i^K\}_{i=1}^I$ have the property that

$$\phi_i^K(\mathbf{x}_j^K) = \delta_{ij},$$

  hence they form a set of local polynomials basis functions on K
- Estimates of interpolation errors will depend on the properties of the affine transformation $\mathbf{T}_K$, for which we have the following result

$$\|\mathbf{T}_K\| \le \frac{h_K}{\hat{\rho}}, \quad \|\mathbf{T}_K^{-1}\| \le \frac{\hat{h}}{\rho_K}$$

Finite Element Method XXIII
Finite Element Spaces

- For second–order boundary value problems we need $V_h \subset H^1(\Omega)$; we may thus set $V_h = X_h$, where

$$X_h = \{v_h \in C(\overline{\Omega}) \,|\, v_h|_K \in X_K,\ \forall K \in \mathcal{T}_h\}$$

  Function in X_h thus have to be continuous across element boundaries
- We can define the following interpolation operators:
  - On the reference element $\hat{K}$: $\hat{\Pi} : C(\hat{K}) \to \hat{X}$, $\hat{\Pi}\hat{v} = \sum_{i=1}^I \hat{v}(\hat{\mathbf{x}})\hat{\phi}_i$ with the interpolation conditions $\hat{\Pi}\hat{v}(\hat{\mathbf{x}}_i) = \hat{v}(\hat{\mathbf{x}}_i)$, $i = 1, \ldots, I$
  - On any actual element K: $\Pi_K : C(K) \to X_K$, $\Pi_K v = \sum_{i=1}^I v(\mathbf{x}^K)\phi_i^K$ with the interpolation conditions $\Pi_K v(\mathbf{x}_i^K) = v(\mathbf{x}_i^K)$, $i = 1, \ldots, I$
- The two interpolation operators are related as $\hat{\Pi}\hat{v} = (\Pi_K v) \circ F_K^{-1} = \widehat{\Pi_K v}$, relation that is essential for error analysis.

Finite Element Method XXIV
Quadratures

- When solving the approximate problem in the form

$$\text{find } u_h \in V_h, \quad a_h(u_h, v_h) = f_h(v_h) \quad \forall v_h \in V_h$$

  we need to evaluate integrals

$$a_h(u_h, v_h) = \int_\Omega A_h(x, u_h, v_h)\,dx,\; f_h(v_h) = \int_\Omega F_h(x, v_h)\,dx + \int_\Gamma G_h(x, v_h)\,dx,$$

  where A_h, F_h, and G_h are operators. In most practical situations these integrals cannot be evaluated analytically and approximate approaches need to be used.
- Quadrature is a method to evaluate an integral approximately.
- Gaussian Quadrature seeks to obtain the best numerical estimate of an integral by picking optimal points x_i, $i = 1, \ldots, N$ at which to evaluate the function $f(x)$.
- The fundamental theorem of Gaussian quadrature states that the optimal abscissas of the N-point Gaussian quadrature formulas are precisely the roots of the orthogonal polynomial for the same interval and weighting function

Finite Element Method XXV
Quadratures

- Definition — Let K be a non-empty, Lipschitz, compact subset of $\mathbb{R}^d$. Let $l_q \geq 1$ be an integer. A quadrature on K with l_q points consists of
  - A set of l_q real numbers $\{\omega_1, \ldots, \omega_{l_q}\}$ called quadrature weights
  - A set of l_q points $\{\xi_1, \ldots, \xi_{l_q}\}$ in K called Gauß points or quadrature nodes

  The largest integer k such that $\forall p \in P_k$, $\int_K p(x)\,dx = \sum_{l=1}^{l_q} \omega_l\, p(\xi_l)$ is called the quadrature order and is denoted by k_q
- As regards 1D bounded intervals, the most frequently used quadratures are based on Legendre polynomials which are defined on the interval $(0,1)$ as $\mathcal{L}_k(t) = \frac{1}{k!}\frac{d^k}{dt^k}(t^2 - t)^k$, $k \geq 0$. Note that they are orthogonal on $(0,1)$ with the weight $W = 1$.

Finite Element Method XXVI
Quadratures

- Theorem — Let $l_q \geq 1$, denote by $\xi_1, \ldots, \xi_{l_q}$ the l_q roots of the Legendre polynomial $\mathcal{L}_{l_q}(x)$ and set $\omega_l = \int_0^1 \prod_{\substack{j=1 \\ j \neq l}}^{l_q} \frac{t - \xi_j}{\xi_l - \xi_j}\,dt$. Then $\{\xi_1, \ldots, \xi_{l_q}, \omega_1, \ldots, \omega_{l_q}\}$ is a quadrature of order $k_q = 2l_q - 1$ on $[0,1]$

  Proof — Let $\{\mathcal{L}_1, \ldots, \mathcal{L}_{l_q}\}$ be the set of Lagrange polynomials associated with the Gauß points $\{\xi_1, \ldots, \xi_{l_q}\}$. Then $\omega_l = \int_0^1 \mathcal{L}_l(t)\,dt$, $1 \leq l \leq l_q$
  - *when $p(x)$ is a polynomial of degree less than l_q, we integrate both sides of the identity $p(t) = \sum_{l=1}^{l_q} p(\xi_l)\mathcal{L}_l(t)\,dx$, $\forall t \in [0,1]$ and deduce that the quadrature is exact for $p(x)$*
  - *when the polynomial $p(x)$ has degree less than $2l_q$ we write it in the form $p(x) = q(x)\mathcal{L}_{l_q}(x) + r(x)$, where both $q(x)$ and $r(x)$ are polynomials of degree less than l_q; owing to orthogonality of the Legendre polynomials, we conclude*

$$\int_0^1 p(t)\,dt = \int_0^1 r(t)\,dt = \sum_{l=1}^{l_q} \omega_l r(\xi_l) = \sum_{l=1}^{l_q} \omega_l\, p(\xi_l),$$

    since the points ξ_l are also roots of $\mathcal{L}_{l_q}$

Finite Element Method XXVII
Quadratures

- The Gauß points and the weights are available in closed form only for low order quadratures; for higher order quadratures they have to be determined approximately, but are readily available in tables
- Generalization to 2D and 3D is straightforward; the Gauß points and weights are again available in tables
- When assembling FEM matrices, the following type need to be evaluated

$$\int_\Omega \phi(x)\,dx = \sum_{K \in \mathcal{T}_h} \int_K \phi(x)\,dx$$

  Since the mapping $F_K : \hat{K} \to K$ is smooth, the change of variables $x = F_K(\hat{x})$ yields

$$\int_K \phi(x)\,dx = \int_{\hat{K}} \phi(F_K(\hat{x}))\det(J_K(\hat{x}))\,d\hat{x},$$

  where $J_K(\hat{x}) = \frac{\partial F_K(\hat{x})}{\partial \hat{x}}$ is the Jacobian matrix of F_K at $\hat{x}$.

FINITE ELEMENT METHOD XXVIII QUADRATURES

- Thus, the integrals can be evaluated over the reference elements; using a quadrature with l_q Gauß points and weights on $\hat{K}$ we obtain

$$\int_K \phi(x)\,dx \approx \sum_{l=1}^{l_q} \omega_l \det(J_K(\xi_l))\phi(F_K(\xi_l)) = \sum_{l=1}^{l_q} \omega_{lK}\phi(\xi_{lK}),$$

  where we set $\omega_{lK} = \omega_l \det(J_K(\xi_l))$ and $\xi_{lK} = F_K(\xi_l)$
- It can be shown that the quadrature $\{\xi_{1K},\dots,\xi_{l_qK},\omega_{1K},\dots,\omega_{l_qK}\}$ enjoy similar properties regarding accuracy as the quadratures defined on K