WELL–POSEDNESS OF PROBLEMS I

- Consider the following generic problem

$$\mathcal{L} u = f,$$

  where $\mathcal{L} : X \rightarrow Y$, $u \in X$, $f \in Y$ and X, Y are two Banach spaces

- We say that the above problem is well-posed (according to Hadamard) iff the following two conditions are satisfied:
  - there exists one and only one solution to this problem,
  - it is characterized by the following a priori estimate

$$\exists c > 0, \ \ \forall f \in Y, \ \ \|u\|_X \leq c \|f\|_Y,$$

    which means that the solution should continuously depend on the data for the problem

- Note that many meaningful and important problems in physics and engineering are in fact not well–posed (hence, they are ill–posed)
- It is expected that if the continuous problem is well–posed, such should be its discrete approximation

WELL–POSEDNESS OF PROBLEMS II

- Well–posedness in the context of the Lax–Milgram Lemma for a problem

$$u \in V, \quad a(u,v) = l(v), \quad \forall v \in V$$

with $a(v,v) \geq \alpha \|v\|^2$, $\forall v \in V$. Consider V^* as the dual space of V; then

$$\|l\|_{V^*} = \sup_{v \in V} \frac{l(v)}{\|v\|_V} = \sup_{v \in V} \frac{a(u,v)}{\|v\|_V} \geq \frac{a(u,u)}{\|u\|_V} \geq \alpha \|u\|_V$$

Thus $\|u\|_V \leq \frac{1}{\alpha} \|l\|_{V^*}$ (α is known as the coercivity constant)

- Conforming and consistent FEM approximations preserve this property, i.e., for $V_h \in V$, the corresponding discrete problem

$$u_h \in V_h, \quad a(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h$$

is also well–posed (in the sense of Hadamard).

- Can $\alpha \to 0$? What will happen in such case?

WELL–POSEDNESS OF PROBLEMS III

- Consider an advection–diffusion problem

$$-\nu\Delta u + \beta\nabla u = f \quad \text{in } \Omega$$

  where $\nu > 0$ is the diffusion coefficient and $\beta : \Omega \to \mathbb{R}^d$ is the advection velocity

- Defining the bilinear form as

$$a(u,v) = \int_\Omega [\nu\nabla u \cdot \nabla v + v(\beta\nabla u)]\, dx$$

  one can show that $\frac{|a|}{\alpha} = O\left(\frac{\|\beta\|_{L_\infty}}{\nu}\right)$, hence $\nu \to 0$ implies less of coercivity

- Such problems, even though they remain formally well–posed, are very difficult to solve using standard approaches
- Coercivity is entirely lost when $\nu = 0$, i.e. for first–order PDEs
- This a different, more general treatment, is required.

FEM FOR FIRST–ORDER PDES I

- Consider a generic first–order PDE

$$\begin{cases} u'(x) = f(x), \;\; \text{in} \;\; \Omega = (0,1) \\ u(0) = 0 \end{cases}$$

- Defining $U = \{v \in H^1(\Omega), v(0) = 0\}$, a possible weak formulation is

$$u \in U, \quad a(u,v) = l(v), \quad \forall v \in V$$

  where $a : U \times V \longrightarrow \mathbb{R}$ is defined as $a(u,v) = \int_\Omega u'v\,dx$ and $V = L_2(\Omega)$

- Now the solution and trial spaces, U and V respectively, are not the same, so the Lax–Milgram Lemma does not apply anymore.

- Existence and uniqueness of solutions of the above problem is addressed by the Generalized Lax–Milgram Lemma [a. k. a. the Ladyzhanskaya–Babuška–Brebbia (LBB), or Banach–Nečas–Babuška (BNB) Theorem]

FEM FOR FIRST–ORDER PDEs II
GENERALIZED LAX–MILGRAM LEMMA

- Theorem — Let U be Banach space and let V be a reflexive Banach space. Let $a : U \times V \to \mathbb{R}$ and $f \in V^*$. Then the problem

$$u \in U, \quad a(u,v) = l(v), \quad \forall v \in V$$

  is well–posed iff

  – $\exists \alpha > 0, \inf_{u \in U} \sup_{v \in V} \frac{a(u,v)}{\|u\|_U \|v\|_V} \geq \alpha$ (the inf–sup condition)

  – $\forall v \in V, \ (\forall u \in U, \ a(u,v) = 0) \ \Rightarrow \ (v = 0)$

  Moreover, we have $\|u\|_U \leq \frac{1}{\alpha} \|f\|_{V^*}, \forall f \in V^*$

- *Proof (outline) — from Banach's closed–range theorem* $\|Au\|_{V^*} \geq \alpha \|u\|_U$*; then*

$$\|Au\|_{V^*} = \sup_{v \in V} \frac{\langle Au, v \rangle}{\|v\|_V} = \sup_{v \in V} \frac{a(u,v)}{\|v\|_V} \geq \alpha \|u\|_U$$

FEM for First–Order PDEs III

- Now return to the original problem, i.e.,

$$u \in U, \quad a(u,v) = l(v), \quad \forall v \in V$$

with $a(u,v) = \int_\Omega u' v \, dx$, $U = \{v \in H^1(\Omega), v(0) = 0\}$ and $V = L_2(\Omega)$

- To examine well–posedness, consider the Generalized Lax–Milgram Lemma

$$\inf_{u \in U} \sup_{v \in L_2(\Omega)} \frac{a(u,v)}{\|u\|_{H^1} \|v\|_{L_2}} = \inf_{u \in U} \frac{\sqrt{\int_0^1 (u')^2 \, dx}}{\|u\|_{H^1}}$$

Using now the Poincaré inequality, i.e., $c\|v\|_{L_2} \leq \|\nabla v\|_{L_2}$, $\forall v \in H_0^1(0,1)$, we can show that

$$\inf_{u \in U} \frac{\sqrt{\int_0^1 (u')^2 \, dx}}{\|u\|_{H^1}} = \inf_{u \in U} \sqrt{\frac{\int_0^1 (u')^2 \, dx}{\int_0^1 u^2 + (u')^2 \, dx}} \geq \sqrt{\frac{2}{3}}$$

- Hence our problem, with the choice of the spaces U and V, is well–posed

FEM for First–Order PDEs IV

- For the discrete problem to be well-posed, the inf–sup condition must also hold in the discrete sense
- For our example 1D first–order equation, consider the corresponding discrete problem

$$u_h \in U_h, \quad a(u_h, v_h) = (f, v_h)_{L_2}, \quad \forall v_h \in U_h,$$

where $U_h = \{u_h \in C^0(\overline{\Omega}); u_h|_{[x_i, x_{i+1}]} \in P_1,\ u_h(0) = 0\}$

- It can be shown that

$$c_1 h \leq \inf_{u_h \in U_h} \sup_{v_h \in U_h} \frac{a(u_h, v_h)}{\|u_h\|_{H^1} \|v_h\|_{L_2}} \leq c_2 h$$

Hence, as $h \to 0$, we have $\alpha = c_1 h \to 0$ and loss of coercivity occurs

- This will not happen when the discrete problem has the form

$$u_h \in U_h, \quad a(u_h, v_h) = (f, v_h)_{L_2}, \quad \forall v_h \in V_h,$$

where U_h is defined as above and $V_h = \{v_h|_{[x_i, x_{i+1}]} \in P_0\}$

FEM for First–Order PDEs V
Least–Squares Approach

- It is often possible to convert a given problem to a form that can be treated using the Galerkin approach;
- To illustrate this approach consider an arbitrary problem in $\mathbb{R}^N$ $AU = F$, where A is a square invertible matrix with no special properties; the corresponding weak formulation will be

$$U \in \mathbb{R}^N, \quad (AU, AV)_N = (F, AV)_N, \quad \forall V \in \mathbb{R}^N,$$

  where $(\cdot,\cdot)$ is an inner product in $\mathbb{R}^N$; Note that this problem is equivalent to $A^T AU = A^T F$, where the matrix $A^T A$ is now symmetric and positive–definite
- Consider the functional $\mathcal{J}(V) = \frac{1}{2}(AV, AV)_N - (F, AV)_N$. The above problem can be therefore recast as the following optimization problem

$$\text{find } U \in \mathbb{R}^N \quad \text{ST} \quad \mathcal{J}(U) = \inf_{V \in \mathbb{R}^N} \mathcal{J}(V)$$

FEM for First–Order PDEs VI
Least–Squares Approach

- Since $\|AV-F\|_N^2 = \mathcal{J}(V) + \frac{1}{2}\|F\|_N^2$, minimization of $\mathcal{J}(V)$ is equivalent to minimization of $\|AV-F\|_N^2$, hence the name the least squares method

- In the context of the problem
$$\begin{cases} u'(x) = f(x), & \text{in } \Omega = (0,1) \\ u(0) = 0 \end{cases}$$
the corresponding weak formulation will be
$$u \in U, \quad \tilde{a}(u,v) = (f, v')_{L_2}, \quad \forall v \in U$$
where $\tilde{a}(u,v) = (u',v')_{L_2}$ and $U = \{v \in H^1(\Omega), v(0) = 0\}$, Note that this is formally equivalent to
$$\begin{cases} u''(x) = f'(x), & \text{in } \Omega = (0,1) \\ u(0) = 0, & u'(1) = f(1) \end{cases}$$

- Note that the matrix A^TA, even though symmetric and positive definite, has a condition number much higher that A

FEM FOR THE STOKES PROBLEM I

- The Stokes Problem often arises in incompressible fluid mechanics ; given $\Omega \subset \mathbb{R}^d$, consider the system of PDE for the unknowns $(\mathbf{u}, p)$

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

  where $\mathbf{u} : \Omega \to \mathbb{R}^d$, $p : \Omega \to \mathbb{R}$ and $\mathbf{f} : \Omega \to \mathbb{R}^d$

- Problems encountered in reality are usually time–dependent ($\frac{\partial \mathbf{u}}{\partial t}$) and nonlinear $[(\mathbf{u} \cdot \nabla)\mathbf{u}]$ — the Navier–Stokes system

- The unknowns $(\mathbf{u}, p)$ have different properties (in terms of smoothness) and special care must be exercised when constructing the appropriate Finite Element interpolations; two approaches are commonly used:
  - mixed formulations
  - constrained formulations

FEM for the Stokes Problem II
Mixed Formulation

- Introduce test functions $\mathbf{v} \in [H_0^1(\Omega)]^d$ and $q \in L_{2(0)}(\Omega) = \{w \in L_2(\Omega);\ \int_\Omega w\, d\Omega = 0\}$; multiplying the first equation by v and the second by q, integrating over Ω and integrating by parts we obtain

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}\, d\Omega - \int_\Omega p \nabla \cdot \mathbf{v}\, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\Omega,$$

$$\int_\Omega q \nabla \cdot \mathbf{u}\, d\Omega = 0$$

- Given $\mathbf{f} \in [H^{-1}(\Omega)]^d$, the following mixed formulation of the weak form is obtained

$$\text{find } \mathbf{u} \in [H_0^1(\Omega)]^d,\ p \in L_{2(0)}(\Omega),\ ST \begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = f(\mathbf{v}), & \forall \mathbf{v} \in [H_0^1(\Omega)]^d \\ b(\mathbf{u},q) = 0 & \forall q \in L_{2(0)}(\Omega) \end{cases}$$

  where $a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}\, d\Omega$ and $b(\mathbf{v},p) = \int_\Omega p \nabla \cdot \mathbf{v}\, d\Omega$

- This weak problem can be shown to be well–posed (unique and bounded solutions exist)

FEM FOR THE STOKES PROBLEM III
MIXED FORMULATION

- Using the discrete approximations of the function spaces $X_h \subset [H_0^1(\Omega)]^d$ and $M_h \subset L_{2(0)}(\Omega)$, we obtain the following discrete problem

$$\text{find } \mathbf{u}_h \in X_h,\ p_h \in M_h,\ ST \begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = f(\mathbf{v}_h), & \forall \mathbf{v}_h \in X_h \\ b(\mathbf{u}_h, q_h) = 0 & \forall q_h \in M_h \end{cases}$$

- The discrete problem is well–posed iff the spaces X_h and M_h are compatible, i.e., the following condition is satisfied

$$\exists \beta_h > 0, \quad \inf_{q_h \in M_h} \sup_{\mathbf{v}_h \in X_h} \frac{\int_\Omega q_h \nabla \cdot \mathbf{v}_h \, d\Omega}{\|q_h\|_{L_2} \|\mathbf{v}_h\|_{H^1}} \geq \beta_h$$

  Ideally, the spaces X_h and M_h should be chosen so that the inf–sup constant β_h be independent from h (uniformly bounded from below)

FEM FOR THE STOKES PROBLEM IV MIXED FORMULATION

- If the inf–sup condition is satisfied, an extension of Céa's lemma gives the following error estimates:

$$\|\mathbf{u}-\mathbf{u}_h\|_{[H^1]^d} \leq c_{1h} \inf_{\mathbf{v}_h \in X_h} \|\mathbf{u}-\mathbf{v}_h\|_{[H^1]^d} + c_{2h} \inf_{q_h \in M_h} \|p-q_h\|_{L_2},$$

$$\|p-p_h\|_{L_2} \leq c_{3h} \inf_{\mathbf{v}_h \in X_h} \|\mathbf{u}-\mathbf{v}_h\|_{[H^1]^d} + c_{4h} \inf_{q_h \in M_h} \|p-q_h\|_{L_2},$$

  where (u,p) is the exact solution of the continuous Stokes problem, and

$$c_{1h} = (1+\frac{|a|}{\alpha})(1+\frac{|b|}{\beta_h}), \qquad c_{2h} = \frac{|b|}{\alpha},$$

$$c_{3h} = c_{1h}\frac{|a|}{\beta_h}, \qquad c_{4h} = 1+\frac{|b|}{\beta_h}+c_{2h}\frac{|a|}{\beta_h},$$

  where α and β_h are the coercivity constants of the bilinear forms $a(u,v) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega$ and $b(v,p) = \int_\Omega p\nabla\cdot\mathbf{u}\, d\Omega$

- Note that if $\beta_h \to 0$ when $h \to 0$, this behavior is more damaging for the rate of convergence of pressure p than that of velocity $\mathbf{u}$

FEM for the Stokes Problem V
Mixed Formulation

- Examples of bad (i.e., violating the inf–sup condition) combinations of the spaces X_h and M_h:
  - Q_1 / T_0
  - T_1 / T_1
  - T_1 / T_0
- Examples of good (i.e., satisfying the inf–sup condition) combinations of the spaces X_h and M_h:
  - T_1–bubble / T_1 (the "T_1–bubble" finite element is characterized by an additional node at the barycenter of the element)
  - T_2 / T_1
- Note that the finite element spaces T_k, $k > 0$, are also denoted P_k