

WELL-POSEDNESS OF PROBLEMS I

- Consider the following generic problem

$$\mathcal{L}u = f,$$

where $\mathcal{L} : X \rightarrow Y$, $u \in X$, $f \in Y$ and X, Y are two Banach spaces

- We say that the above problem is **well-posed** (according to Hadamard) iff the following two conditions are satisfied:
 - there exists one and only one solution to this problem,
 - it is characterized by the following **a priori** estimate

$$\exists c > 0, \forall f \in Y, \|u\|_X \leq c\|f\|_Y,$$

which means that the solution should **continuously depend on the data for the problem**

- Note that many meaningful and important problems in physics and engineering are in fact not well-posed (hence, they are ill-posed)
- It is expected that if the continuous problem is well-posed, such should be its discrete approximation

WELL-POSEDNESS OF PROBLEMS II

- Well-posedness in the context of the Lax–Milgram Lemma for a problem

$$u \in V, \quad a(u, v) = l(v), \quad \forall v \in V$$

with $a(v, v) \geq \alpha\|v\|^2$, $\forall v \in V$. Consider V^* as the **dual space** of V ; then

$$\|l\|_{V^*} = \sup_{v \in V} \frac{l(v)}{\|v\|_V} = \sup_{v \in V} \frac{a(u, v)}{\|v\|_V} \geq \frac{a(u, u)}{\|u\|_V} \geq \alpha\|u\|_V$$

Thus $\|u\|_V \leq \frac{1}{\alpha}\|l\|_{V^*}$ (α is known as the **coercivity constant**)

- Conforming and consistent FEM approximations preserve this property, i.e., for $V_h \in V$, the corresponding discrete problem

$$u_h \in V_h, \quad a(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h$$

is also well-posed (in the sense of Hadamard).

- Can $\alpha \rightarrow 0$? What will happen in such case?

WELL-POSEDNESS OF PROBLEMS III

- Consider an advection–diffusion problem

$$-v\Delta u + \beta \nabla u = f \text{ in } \Omega$$

where $v > 0$ is the diffusion coefficient and $\beta : \Omega \rightarrow \mathbb{R}^d$ is the advection velocity

- Defining the bilinear form as

$$a(u, v) = \int_{\Omega} [v \nabla u \cdot \nabla v + v(\beta \nabla u)] dx$$

one can show that $\frac{|a|}{\alpha} = O\left(\frac{\|\beta\|_{L^\infty}}{v}\right)$, hence $v \rightarrow 0$ implies **less of coercivity**

- Such problems, even though they remain formally well-posed, are very difficult to solve using standard approaches
- Coercivity is entirely lost when $v = 0$, i.e. for first-order PDEs
- This a different, more general treatment, is required.

FEM FOR FIRST-ORDER PDES I

- Consider a generic first-order PDE

$$\begin{cases} u'(x) = f(x), & \text{in } \Omega = (0, 1) \\ u(0) = 0 \end{cases}$$

- Defining $U = \{v \in H^1(\Omega), v(0) = 0\}$, a possible weak formulation is

$$u \in U, \quad a(u, v) = l(v), \quad \forall v \in V$$

where $a : U \times V \rightarrow \mathbb{R}$ is defined as $a(u, v) = \int_{\Omega} u'v dx$ and $V = L_2(\Omega)$

- Now the solution and trial spaces, U and V respectively, are not the same, so the Lax–Milgram Lemma does not apply anymore.
- Existence and uniqueness of solutions of the above problem is addressed by the **Generalized Lax–Milgram Lemma** [a. k. a. the Ladyzhanskaya–Babuška–Brebbia (LBB), or Banach–Nečas–Babuška (BNB) Theorem]

FEM FOR FIRST-ORDER PDES II GENERALIZED LAX-MILGRAM LEMMA

- Theorem — Let U be Banach space and let V be a reflexive Banach space. Let $a : U \times V \rightarrow \mathbb{R}$ and $f \in V^*$. Then the problem

$$u \in U, \quad a(u, v) = l(v), \quad \forall v \in V$$

is well-posed iff

- $\exists \alpha > 0, \inf_{u \in U} \sup_{v \in V} \frac{a(u, v)}{\|u\|_U \|v\|_V} \geq \alpha$ (the inf-sup condition)
- $\forall v \in V, (a(u, v) = 0) \Rightarrow (v = 0)$

Moreover, we have $\|u\|_U \leq \frac{1}{\alpha} \|f\|_{V^*}, \forall f \in V^*$

- Proof (outline) — from Banach's closed-range theorem $\|Au\|_{V^*} \geq \alpha \|u\|_U$; then

$$\|Au\|_{V^*} = \sup_{v \in V} \frac{\langle Au, v \rangle}{\|v\|_V} = \sup_{v \in V} \frac{a(u, v)}{\|v\|_V} \geq \alpha \|u\|_U$$

FEM FOR FIRST-ORDER PDES III

- Now return to the original problem, i.e.,

$$u \in U, \quad a(u, v) = l(v), \quad \forall v \in V$$

with $a(u, v) = \int_{\Omega} u' v dx, U = \{v \in H^1(\Omega), v(0) = 0\}$ and $V = L_2(\Omega)$

- To examine well-posedness, consider the Generalized Lax-Milgram Lemma

$$\inf_{u \in U} \sup_{v \in L_2(\Omega)} \frac{a(u, v)}{\|u\|_{H^1} \|v\|_{L_2}} = \inf_{u \in U} \frac{\sqrt{\int_0^1 (u')^2 dx}}{\|u\|_{H^1}}$$

Using now the Poincaré inequality, i.e., $c\|v\|_{L_2} \leq \|\nabla v\|_{L_2}, \forall v \in H_0^1(0, 1)$, we can show that

$$\inf_{u \in U} \frac{\sqrt{\int_0^1 (u')^2 dx}}{\|u\|_{H^1}} = \inf_{u \in U} \sqrt{\frac{\int_0^1 (u')^2 dx}{\int_0^1 u^2 + (u')^2 dx}} \geq \sqrt{\frac{2}{3}}$$

- Hence our problem, with the choice of the spaces U and V , is **well-posed**

FEM FOR FIRST-ORDER PDES IV

- For the discrete problem to be well-posed, the **inf-sup** condition must also hold in the discrete sense
- For our example 1D first-order equation, consider the corresponding discrete problem

$$u_h \in U_h, \quad a(u_h, v_h) = (f, v_h)_{L_2}, \quad \forall v_h \in U_h,$$

where $U_h = \{u_h \in C^0(\bar{\Omega}); u_h|_{[x_i, x_{i+1}]} \in P_1, u_h(0) = 0\}$

- It can be shown that

$$c_1 h \leq \inf_{u_h \in U_h} \sup_{v_h \in U_h} \frac{a(u_h, v_h)}{\|u_h\|_{H^1} \|v_h\|_{L_2}} \leq c_2 h$$

Hence, as $h \rightarrow 0$, we have $\alpha = c_1 h \rightarrow 0$ and loss of coercivity occurs

- This will not happen when the discrete problem has the form

$$u_h \in U_h, \quad a(u_h, v_h) = (f, v_h)_{L_2}, \quad \forall v_h \in V_h,$$

where U_h is defined as above and $V_h = \{v_h|_{[x_i, x_{i+1}]} \in P_0\}$

FEM FOR FIRST-ORDER PDES V LEAST-SQUARES APPROACH

- It is often possible to convert a given problem to a form that can be treated using the Galerkin approach;
- To illustrate this approach consider an arbitrary problem in \mathbb{R}^N $AU = F$, where A is a square invertible matrix with no special properties; the corresponding weak formulation will be

$$U \in \mathbb{R}^N, \quad (AU, AV)_N = (F, AV)_N, \quad \forall V \in \mathbb{R}^N,$$

where (\cdot, \cdot) is an inner product in \mathbb{R}^N ; Note that this problem is equivalent to $A^T AU = A^T F$, where the matrix $A^T A$ is now **symmetric and positive-definite**

- Consider the functional $J(V) = \frac{1}{2}(AV, AV)_N - (F, AV)_N$. The above problem can be therefore recast as the following optimization problem

$$\text{find } U \in \mathbb{R}^N \quad \text{ST} \quad J(U) = \inf_{V \in \mathbb{R}^N} J(V)$$

FEM FOR FIRST-ORDER PDES VI LEAST-SQUARES APPROACH

- Since $\|AV - F\|_N^2 = J(V) + \frac{1}{2}\|F\|_N^2$, minimization of $J(V)$ is equivalent to minimization of $\|AV - F\|_N^2$, hence the name **the least squares method**

- In the context of the problem

$$\begin{cases} u'(x) = f(x), & \text{in } \Omega = (0, 1) \\ u(0) = 0 \end{cases}$$

the corresponding weak formulation will be

$$u \in U, \quad \tilde{a}(u, v) = (f, v')_{L_2}, \quad \forall v \in U$$

where $\tilde{a}(u, v) = (u', v')_{L_2}$ and $U = \{v \in H^1(\Omega), v(0) = 0\}$, Note that this is formally equivalent to

$$\begin{cases} u''(x) = f'(x), & \text{in } \Omega = (0, 1) \\ u(0) = 0, \quad u'(1) = f(1) \end{cases}$$

- Note that the matrix $A^T A$, even though symmetric and positive definite, has a condition number much higher than A

FEM FOR THE STOKES PROBLEM I

- The **Stokes Problem** often arises in **incompressible fluid mechanics**; given $\Omega \subset \mathbb{R}^d$, consider the system of PDE for the unknowns (\mathbf{u}, p)

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

where $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$, $p : \Omega \rightarrow \mathbb{R}$ and $\mathbf{f} : \Omega \rightarrow \mathbb{R}^d$

- Problems encountered in reality are usually time-dependent ($\frac{\partial \mathbf{u}}{\partial t}$) and nonlinear $[(\mathbf{u} \cdot \nabla)\mathbf{u}]$ — the **Navier–Stokes system**
- The unknowns (\mathbf{u}, p) have different properties (in terms of smoothness) and special care must be exercised when constructing the appropriate Finite Element interpolations; two approaches are commonly used:
 - mixed formulations
 - constrained formulations

FEM FOR THE STOKES PROBLEM II MIXED FORMULATION

- Introduce test functions $\mathbf{v} \in [H_0^1(\Omega)]^d$ and $q \in L_{2(0)}(\Omega) = \{w \in L_2(\Omega); \int_{\Omega} w d\Omega = 0\}$; multiplying the first equation by \mathbf{v} and the second by q , integrating over Ω and integrating by parts we obtain

$$\begin{aligned} \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} d\Omega - \int_{\Omega} p \nabla \cdot \mathbf{v} d\Omega &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\Omega, \\ \int_{\Omega} q \nabla \cdot \mathbf{u} d\Omega &= 0 \end{aligned}$$

- Given $\mathbf{f} \in [H^{-1}(\Omega)]^d$, the following **mixed formulation of the weak form** is obtained

$$\text{find } \mathbf{u} \in [H_0^1(\Omega)]^d, p \in L_{2(0)}(\Omega), \text{ ST } \begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = f(\mathbf{v}), & \forall \mathbf{v} \in [H_0^1(\Omega)]^d \\ b(\mathbf{u}, q) = 0 & \forall q \in L_{2(0)}(\Omega) \end{cases}$$

where $a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} d\Omega$ and $b(\mathbf{v}, p) = \int_{\Omega} p \nabla \cdot \mathbf{v} d\Omega$

- This weak problem can be shown to be well-posed (unique and bounded solutions exist)

FEM FOR THE STOKES PROBLEM III MIXED FORMULATION

- Using the discrete approximations of the function spaces $X_h \subset [H_0^1(\Omega)]^d$ and $M_h \subset L_{2(0)}(\Omega)$, we obtain the following discrete problem

$$\text{find } \mathbf{u}_h \in X_h, p_h \in M_h, \text{ ST } \begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = f(\mathbf{v}_h), & \forall \mathbf{v}_h \in X_h \\ b(\mathbf{u}_h, q_h) = 0 & \forall q_h \in M_h \end{cases}$$

- The discrete problem is well-posed iff the spaces X_h and M_h are compatible, i.e., the following condition is satisfied

$$\exists \beta_h > 0, \quad \inf_{q_h \in M_h} \sup_{\mathbf{v}_h \in X_h} \frac{\int_{\Omega} q_h \nabla \cdot \mathbf{v}_h d\Omega}{\|q_h\|_{L_2} \|\mathbf{v}_h\|_{H^1}} \geq \beta_h$$

Ideally, the spaces X_h and M_h should be chosen so that the inf-sup constant β_h be independent from h (uniformly bounded from below)

FEM FOR THE STOKES PROBLEM IV MIXED FORMULATION

- If the inf-sup condition is satisfied, an extension of C  a's lemma gives the following error estimates:

$$\|\mathbf{u} - \mathbf{u}_h\|_{[H^1]^d} \leq c_{1h} \inf_{\mathbf{v}_h \in X_h} \|\mathbf{u} - \mathbf{v}_h\|_{[H^1]^d} + c_{2h} \inf_{q_h \in M_h} \|p - q_h\|_{L_2},$$

$$\|p - p_h\|_{L_2} \leq c_{3h} \inf_{\mathbf{v}_h \in X_h} \|\mathbf{u} - \mathbf{v}_h\|_{[H^1]^d} + c_{4h} \inf_{q_h \in M_h} \|p - q_h\|_{L_2},$$

where (u, p) is the exact solution of the continuous Stokes problem, and

$$c_{1h} = \left(1 + \frac{|a|}{\alpha}\right) \left(1 + \frac{|b|}{\beta_h}\right), \quad c_{2h} = \frac{|b|}{\alpha},$$

$$c_{3h} = c_{1h} \frac{|a|}{\beta_h}, \quad c_{4h} = 1 + \frac{|b|}{\beta_h} + c_{2h} \frac{|a|}{\beta_h},$$

where α and β_h are the **coercivity constants** of the bilinear forms

$$a(u, v) = \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} d\Omega \text{ and } b(v, p) = \int_{\Omega} p \nabla \cdot \mathbf{u} d\Omega$$

- Note that if $\beta_h \rightarrow 0$ when $h \rightarrow 0$, this behavior is more damaging for the rate of convergence of pressure p than that of velocity \mathbf{u}

FEM FOR THE STOKES PROBLEM V MIXED FORMULATION

- Examples of **bad** (i.e., violating the inf-sup condition) combinations of the spaces X_h and M_h :
 - Q_1 / T_0
 - T_1 / T_1
 - T_1 / T_0
- Examples of **good** (i.e., satisfying the inf-sup condition) combinations of the spaces X_h and M_h :
 - T_1 -bubble / T_1 (the “ T_1 -bubble” finite element is characterized by an additional node at the barycenter of the element)
 - T_2 / T_1
- Note that the finite element spaces T_k , $k > 0$, are also denoted P_k