Part II

Review of Sobolev Spaces

Sobolev Spaces — Weak Derivatives I

- Given $\mathbb{R}^d$, define a multi–index α as an ordered collection of integers $\alpha = (\alpha_1, \dots, \alpha_d)$, such that its length is given by $|\alpha| = \sum_{i=1}^d \alpha_i$
- If v is an m–times differentiable function, then for any α with $|\alpha| \le m$ the derivative can be expressed as

$$D^\alpha v(\mathbf{x}) = \frac{\partial^{|\alpha|} v(\mathbf{x})}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$$

- integration by parts formula — given an open domain $\Omega \subseteq \mathbb{R}^d$ and $v \in C^m(\Omega)$, $\phi \in C_0^\infty(\Omega)$ with $|\alpha| \le m$

$$\int_\Omega v(\mathbf{x}) D^\alpha \phi(\mathbf{x})\, dx = (-1)^{|\alpha|} \int_\Omega D^\alpha v(\mathbf{x}) \phi(\mathbf{x})\, dx$$

- Definition — given $v, w \in L^1(\Omega)$, w is called a weak α–th derivative of v iff

$$\int_\Omega v(\mathbf{x}) D^\alpha \phi(\mathbf{x})\, dx = (-1)^{|\alpha|} \int_\Omega w(\mathbf{x}) \phi(\mathbf{x})\, dx, \quad \forall \phi \in C_0^\infty(\Omega)$$

Sobolev Spaces — Weak Derivatives II

- A weak derivative, if it exists, is defined up to a set of measure zero
- In the previous formula, if $v \in C^m(\Omega)$, then for each α with $|\alpha| \le m$ the classical partial derivative $D^\alpha v$ is also the weak α–th partial derivative of v; usually the same symbol is used to denote the two kinds of derivative
- Thus, the weak derivative can be regarded as an extension of the classical derivative to functions which are not differentiable in the classical sense.
- Examples:
  - The absolute value function $v(x) = |x|$ is continuous at $x = 0$, but not differentiable in the classical sense. Nevertheless, its first–order weak derivative exists and is given by ($c_0 \in \mathbb{R}$ is arbitrary)

$$w(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0, \\ c_0, & x = 0. \end{cases}$$

Sobolev Spaces — Weak Derivatives III

- Examples (cont'd):
  - Functions with jump discontinuities, e.g.,

$$v(x) = \begin{cases} -1, & -1 < x < 0, \\ c_0, & x = 0, \\ 1, & 0 < x < 1. \end{cases}$$

    are not weakly differentiable
  - More generally, assume that $v \in C[a,b]$ is piecewise continuously differentiable, i.e., there exists a partition of the interval $a = x_0 < x_1 \cdots < x_n = b$ such that $v \in C^1[x_{i-1}, x_i]$, $1 \le i \le n$. Then the first–order weak derivative of v is

$$w(x) = \begin{cases} v'(x), & x \in \bigcup_{i=1}^n (x_{i-1}, x_i), \\ \text{arbitrary}, & x = x_i, \ , 1 \le i \le n. \end{cases}$$

    Note that the second–order weak derivative does not exist.
- Weak derivative share many properties with classical derivatives, such as linearity, chain rule, differentiation of products, etc.

Sobolev Spaces — Domain Boundaries I

- Sobolev spaces require some regularity of the boundary $\partial\Omega$ of the domain Ω
- Definition — Let Ω be open and bounded in $\mathbb{R}^d$ and let V denote a function space on $\mathbb{R}^{d-1}$. We say that $\partial\Omega$ is of class V is for each point $\mathbf{x}_0 \in \partial\Omega$ there exists an $r > 0$ and a function $g \in V$ such that upon transformation of the coordinate system if necessary, we have

$$\Omega \bigcap B(\mathbf{x}_0, r) = \{\mathbf{x} \in B(\mathbf{x}_0, r) | x_d > g(x1, \ldots, x_{d-1})\},$$

  where $B(\mathbf{x}_0, r)$ denoted the d–dimensional ball centered at x_0 with radius r. In particular, when V consists of:
  - Lipschitz continuous functions, we say Ω is a Lipschitz domain,
  - C^k functions, we say Ω is a C^k domain,
  - $C^{k,\alpha}$, $(0 < \alpha \leq 1)$ functions, we say Ω is a Hölder boundary of class $C^{k,\alpha}$
- Most domains arising in physical and engineering applications are Lipschitz (may have sharp corners, but not cusps).

Sobolev Spaces — Integer Order I

- Definition — Let k be a non–negative integer, $p \in [1, \infty]$. The Sobolev space $W^{k,p}(\Omega)$ is the set of all the functions v such that for each multi–index α with $|\alpha| \leq k$, the α–th weak derivative $D^\alpha v$ exists and $D^\alpha v \in L^p(\Omega)$. The norm in the space $W^{k,p}(\Omega)$ is defined as

$$\|v\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{1/p}, & 1 \leq p < \infty, \\ \max_{|\alpha| \leq k} \|D^\alpha v\|_{L^\infty(\Omega)}, & p = \infty \end{cases}$$

  When $p = 2$, we write $H^k(\Omega) = W^{k,2}(\Omega)$. Obviously, we have $H^0(\Omega) = L^2(\Omega)$
- The Sobolev space $W^{k,p}(\Omega)$ is a Banach space, while $H^k(\Omega)$ is a Hilbert space with the inner product given by

$$(u, v)_k = \int_\Omega \sum_{|\alpha| \leq k} D^\alpha u(\mathbf{x}) D^\alpha v(\mathbf{x})\, dx, \quad u, v \in H^k(\Omega)$$

Sobolev Spaces — Real Order I

- It is possible to extend the definition of Sobolev spaces to ant real order (including negative)
- Definition — Assume that $p \in [1, \infty)$ and let $s = k + \sigma$ with $k \geq 0$ an integer and $\sigma \in (0, 1)$. Then we define the Sobolev space

$$W^{s,p}(\Omega) = \left\{ v \in W^{k,p}(\Omega) \mid \frac{|D^\alpha v(\mathbf{x}) - D^\alpha v(\mathbf{y})|}{\|\mathbf{x} - \mathbf{y}\|^{\sigma + d/p}} \in L^p(\Omega \times \Omega),\ \forall \alpha : |\alpha| = k \right\}$$

  with the norm

$$\|v\|_{W^{s,p}(\Omega)} = \left(\|v\|_{W^{k,p}(\Omega)} + \sum_{|\alpha| = k} \int_{\Omega \times \Omega} \frac{|D^\alpha v(\mathbf{x}) - D^\alpha v(\mathbf{y})|^p}{\|\mathbf{x} - \mathbf{y}\|^{\sigma p + d}}\, dxdy \right)^{1/p}$$

- When $p = 2$, $H^s(\Omega) = W^{s,2}(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{s,\Omega} = (u, v)_{k,\Omega} + \sum_{|\alpha| = k} \int_{\Omega \times \Omega} \frac{(D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y}))(D^\alpha v(\mathbf{x}) - D^\alpha v(\mathbf{y}))}{\|\mathbf{x} - \mathbf{y}\|^{2\sigma + d}}\, dxdy$$

Sobolev Spaces — Real Order II

- Definition — Let $s \geq 0$. We define $W_0^{s,p}(\Omega)$ to be the closure of the space $C_0^\infty(\Omega)$ in $W^{s,p}(\Omega)$. When $p = 2$, we have a Hilbert space $H_0^s(\Omega) = W_0^{s,2}(\Omega)$
- Let $s \geq 0$ and $p \in [1, \infty)$ and denote its conjugate exponent p' by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. Then we define $W^{-s,p'}(\Omega)$ to be the dual space of $W_0^{s,p}(\Omega)$. In particular, $H^{-s}(\Omega) = W^{-s,2}(\Omega)$
- Example — any $l \in H^{-1}(\Omega)$ is a bounded linear functional on $H_0^1(\Omega)$; the norm of l s given by (note the connection with the operator norm)

$$\|l\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{l(v)}{\|v\|_{H_0^1(\Omega)}}$$

Sobolev Spaces — Embedding Theorems I

- Let V and W be two Banach spaces with $V \subseteq W$. We say the space V is continuously embedded in W and write $V \hookrightarrow W$, if

$$\|v\|_W \leq c\|v\|_V, \quad \forall v \in V$$

If $V \hookrightarrow W$, the function in V are more smooth than the remaining functions in W.

- There are several embedding theorem ; below we cite one of them. Let $\Omega \subseteq \mathbb{R}^d$ be a non–empty open bounded Lipschitz domain. Then the following statements are valid:
  1. if $k < \frac{d}{p}$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \leq p^*$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{d}{p}$,
  2. if $k = \frac{d}{p}$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q < \infty$,
  3. if $k > \frac{d}{p}$, then $W^{k,p}(\Omega) \hookrightarrow C^{k-[\frac{d}{p}]-1,\beta}(\Omega)$, where

$$\beta = \begin{cases} \left[\frac{d}{p}\right] + 1 - \frac{d}{p}, & \text{if } \frac{d}{p} \neq \text{integer}, \\ \text{any positive number} < 1, & \frac{d}{p} = \text{integer} \end{cases}$$

Sobolev Spaces — Embedding Theorems II

- How to understand embedding theorems ???
- Consider the theorem cited above; what it essentially says is the following:
  - the larger the product kp, the smoother the functions from the space $W^{k,p}(\Omega)$,
  - there is a critical value d (the dimension of the domain Ω) for this product such that if $kp > d$, then a $W^{k,p}(\Omega)$ function (and some of its derivatives) are actually continuous
  - when $kp < d$, a $W^{k,p}(\Omega)$ function belongs to $L^{p^*}(\Omega)$ for an exponent p^* larger than p

Sobolev Spaces — Traces I

- Sobolev spaces are defined through the $L^p(\Omega)$ spaces. Hence, Sobolev functions may not be well–defined on the domain boundary $\partial\Omega$, since it has zero measure in $\mathbb{R}^d$
- It is possible to define a trace operator γ, so that γv represents a generalized boundary value of v (it coincides with $v|_{\partial\Omega}$ if v is a function continuous up to the boundary)
- Theorem — Assume Ω is an open, bounded Lipschitz domain in $\mathbb{R}^d$ and $1 \leq p < \infty$. Then there exists a continuous linear operator $\gamma : W^{1,p}(\Omega) \to L^p(\Omega)$ such that:
  - $\gamma v = v|_{\partial\Omega}$ if $v \in W^{1,p}(\Omega) \bigcap C(\overline{\Omega})$,
  - for some constant $C > 0$, $\|\gamma v\|_{L^p(\partial\Omega)} \leq c\|v\|_{W^{1,p}(\Omega)}$, $\forall v \in W^{1,p}(\Omega)$,
  - The mapping $\gamma : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact

Sobolev Spaces — Fourier Transforms I

- When $\Omega = \mathbb{R}^d$, it is possible to construct Sobolev spaces $H^k(\Omega)$ using Fourier transforms which for $v \in L^1(\mathbb{R}^d)$ are defined as

$$\mathcal{F}(v)(\mathbf{y}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} v(\mathbf{x}) e^{(-i\mathbf{x}\cdot\mathbf{y})}\, dx, \quad \mathcal{F}^{-1}(v)(\mathbf{y}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} v(\mathbf{x}) e^{(i\mathbf{x}\cdot\mathbf{y})}\, dx$$

Note that by the Parseval's theorem we have that

$$\|\mathcal{F} v\|_{L^2(\mathbb{R}^d)} = \|\mathcal{F}^{-1} v\|_{L^2(\mathbb{R}^d)} = \|v\|_{L^2(\mathbb{R}^d)}$$

- Theorem — A function $v \in L^2(\mathbb{R}^d)$ belongs to $H^k(\mathbb{R}^d)$ iff $(1 + |\mathbf{y}|^k)\mathcal{F} v \in L^2(\mathbb{R}^d)$. More over, there exist $c_1, c_2 > 0$ such that

$$c_1\|v\|_{H^k(\mathbb{R}^d)} \leq \|(1 + |\mathbf{y}|^k)\mathcal{F} v\|_{L^2(\mathbb{R}^d)} \leq c_2\|v\|_{H^k(\mathbb{R}^d)}$$

Thus $\|(1 + |\mathbf{y}|^k)\mathcal{F} v\|_{L^2(\mathbb{R}^d)}$ defines an equivalent norm on $H^k(\mathbb{R}^d)$

- Note that by replacing k with an $s \in \mathbb{R}$ we can conveniently define Sobolev spaces of non–integer order
- Replacing $\mathbb{R}^d$ with a periodic domain $[0, 2\pi]^d$ leads to remarkable simplifications