PART III

Weak Formulation of Elliptic Boundary Value Problems

A MODEL BOUNDARY VALUE PROBLEM I

- Assume that Ω is an open bounded set in $\mathbb{R}^d$ and its boundary $\Gamma = \partial\Omega$ is Lipschitz continuous
- Consider an example Boundary Value Problem (the Poisson equation)

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \end{cases}$$

- Given $f \in C(\Omega)$, a classical solution of the above problem is a function $u \in C^2(\Omega) \bigcap C(\overline{\Omega})$ which satisfies the above equation and the boundary conditions pointwise
- Note that existence of such classical solutions for more general problems is hard to show ....
- Introduction of weak solutions allow one to remove some of the high smoothness requirements

A MODEL BOUNDARY VALUE PROBLEM II

- Multiply th equation by a smooth test function $v \in C_0^\infty(\Omega)$ and integrate over the domain Ω, then use integration by parts

$$-\int_\Omega \Delta u v \, dx = \underbrace{\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx}_{\bigstar}$$

  Note that the boundary term $\oint_\Gamma \frac{\partial u}{\partial n} v \, d\sigma = 0$, since $v \equiv 0$ on Γ.
- The equation ($\bigstar$) makes sense for much weaker assumptions:
  - $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, so as the test function we can take any $v \in H_0^1(\Omega)$
  - For the RHS it is enough to assume that $f \in H^{-1}(\Omega) = (H_0^1(\Omega))'$
- Thus, the weak formulation of the boundary value problem becomes

$$u \in H_0^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx, \quad \forall v \in H_0^1(\Omega)$$

A MODEL BOUNDARY VALUE PROBLEM III

- Relation between classical and weak solutions:
  - classical solutions are also weak solutions
  - the converse is not true, unless extra regularity is added
- Set $V = H_0^1(\Omega)$ and define:
  - a bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ such that

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad u, v \in V$$

  - a linear functional $l : V \to \mathbb{R}$ such that

$$l(v) = \int_\Omega f v \, dx, \quad v \in V$$

- Then the weak formulation of the problem is to find $u \in V$ such that

$$a(u,v) = l(v), \quad \forall v \in V$$

A Model Boundary Value Problem IV

- Define the following differential operator A associated with the boundary value problem as
$$A : H_0^1(\Omega) \to H^{-1}(\Omega), \quad \langle Au, v\rangle = a(u,v), \quad \forall u,v \in H_0^1(\Omega),$$
where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, i.e., $l(v) = \langle l, v\rangle$ for $l \in H^{-1}(\Omega)$ and $v \in H_0^1(\Omega)$
- Then the weak formulation of the boundary value problem can be rewritten as a linear operator equation in a dual space
$$Au = l, \quad \text{in } H^{-1}(\Omega)$$
- Thus, the formalism of weak formulation allows one to convert a differential equation to an equality of functionals
- Note that the weak formulation does not explicitly state the boundary conditions (they are incorporated into the definition of the function spaces)
- Weak formulations directly lead to Galerkin–type numerical methods

Lax–Milgram Lemma I

- Theorem — There exists a one–to–one correspondence between linear continuous operators $A : V \to V'$ and continuous bilinear forms $a : V \times V \to \mathbb{R}$ given by the formula
$$\langle Au, v\rangle = a(u,v), \quad \forall u,v \in V$$
Consequently, properties of elliptic boundary value problems defined with A can be studied using the properties of the bilinear form a
- Definition — The operator A (resp. the bilinear form a) is said to be V–elliptic iff $\langle Av, v\rangle \geq \alpha\|v\|_V^2$, $\forall v \in V$ (resp. $a(v,v) \geq \alpha\|v\|_V^2$, $\forall v \in V$)
- Theorem — Assume that K is a non–empty closed subspace of the Hilbert space V, $a : V \times V \to \mathbb{R}$ is bilinear, symmetric, bounded and V–elliptic, $l \in V'$. Let
$$E(v) = \frac{1}{2}a(v,v) - l(v), \quad v \in V$$
Then there exists a unique $u \in K$ such that $E(u) = \inf_{v\in K} E(v)$ which is also the unique solution of the equation
$$u \in K, \quad a(u,v) = l(v), \quad \forall v \in K$$

Lax–Milgram Lemma II

- Note that, in practice, the bilinear form $a(\cdot,\cdot)$ may not necessarily be symmetric; The Lax–Milgram Lemma is essential for proving existence and uniqueness of certain operator equations
- Lax–Milgram Lemma - Assume V is a Hilbert space, $a(\cdot,\cdot)$ is a bounded, V–elliptic bilinear form on V and $l \in V'$. Then there is a unique solution of the problem
$$u \in V, \quad a(u,v) = l(v), \quad \forall v \in V$$
- Illustration — consider the case $V = \mathbb{R}$ and a simple linear equation with the corresponding weak formulation
$$x \in \mathbb{R}, \qquad ax = l,$$
$$x \in \mathbb{R}, \qquad axy = ly, \qquad \forall y \in \mathbb{R}$$
To ensure existence of solutions we need:
  - $0 < a < \infty$, i.e., the bilinear form $a(x,y) = axy$ must be continuous and $\mathbb{R}$–elliptic,
  - $|l| < \infty$, i.e., the linear functional $l(y) = ly$ must be bounded

A Model Boundary Value Problem V Non–Homogeneous Dirichlet BCs

- Consider the problem
$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \Gamma, \end{cases}$$
- Assume that $g \in H^{1/2}(\Gamma)$; since (by the Trace Theorem) $\gamma(H^1(\Omega)) = H^{1/2}(\Gamma)$, we have the existence of a function $G \in H^1(\Omega)$ such that $\gamma G = g$
- Thus, we can set $u = w + G$, where w solves the problem with homogeneous Dirichlet boundary conditions
$$\begin{cases} -\Delta w = f + \Delta G, & \text{in } \Omega, \\ w = 0, & \text{on } \Gamma, \end{cases}$$
- The corresponding weak formulation is
$$w \in H_0^1(\Omega), \quad \int_\Omega \nabla w \cdot \nabla v \, dx = \int_\Omega (fv - \nabla G \cdot \nabla v)\, dx, \quad \forall v \in H_0^1(\Omega)$$
Existence of a solution follows from the Lax–Milgram Lemma.

A Model Boundary Value Problem VI
Non–Homogeneous Neumann BCs

- Consider the Helmholtz equation (the corresponding Poisson equation has nonunique solutions)
$$\begin{cases} -\Delta u + u = f, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = g, & \text{on } \Gamma, \end{cases}$$
Note that the classical solution, if exists, $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$
- The weak formulation is
$$u \in H^1(\Omega), \quad \int_\Omega (\nabla v \cdot \nabla v + uv)\,dx = \int_\Omega fv\,dx + \oint_\Gamma gv d\sigma, \quad \forall v \in H^1(\Omega)$$
- Assuming $V = H^1(\Omega)$ and
$$a(u,v) = \int_\Omega (\nabla v \cdot \nabla v + uv)\,dx,$$
$$l(v) = \int_\Omega fv\,dx + \oint_\Gamma gv d\sigma$$
we can apply the Lax–Milgram lemma which guarantees existence of the weak solutions.

A Model Boundary Value Problem VII
General Elliptic Problems

- Consider a general elliptic boundary value problem
$$\begin{cases} -\partial_j(A_{ij}\partial_i u) + B_i\partial_i u + Cu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D, \\ R_{ij}(\partial_i u)n_j = g, & \text{on } \Gamma_N, \end{cases}$$
where the boundary $\Gamma = \overline{\Gamma}_D \bigcup \overline{\Gamma}_N$ with $\Gamma_D \bigcap \Gamma_N = \emptyset$
- The following assumptions are made regarding the data:
  - $A_{ij}, B_j, C \in L^\infty(\Omega)$
  - $\exists \theta > 0 : A_{ij}\xi_i\xi_j \geq \theta|\boldsymbol{\xi}|^2, \ \ \forall \boldsymbol{\xi} \in \mathbb{R}^d$, a.e. in Ω
  - $f \in L^2(\Omega), g \in L^2(\Gamma_N)$

A Model Boundary Value Problem VIII
General Elliptic Problems

- The weak formulation is obtained by setting $V = H^1_{\Gamma_D}(\Omega)$ and
$$a(u,v) = \int_\Omega (A_{ij}\partial_i u \partial_j v + B_i(\partial_i u)v + Cuv)\,dx,$$
$$l(v) = \int_\Omega fv\,dx + \oint_{\Gamma_N} gv d\sigma$$
- Existence of weak solutions again follows from the application of the Lax–Milgram Lemma

The Galerkin Method I

- The Galerkin Method provides a natural framework for finding finite–dimensional approximation of weak solutions of elliptic boundary value problems
- Assume that $a(\cdot,\cdot)$ is
  - bounded (i.e., $|a(u,v) \leq M\|u\|_V\|v\|_V, \forall u,v \in V$), and
  - V–elliptic (i.e., $a(v,v) \geq c_0\|v\|^2_V, \forall v \in V$)
- Given an N–dimensional subspace $V_N \subseteq V$, consider the problem
$$u_N \in V_N, \quad a(u_N,v) = l(v), \quad \forall v \in V_N$$
- With the above assumptions, Lax–Milgram Lemma guarantees existence of an unique solution u_N

The Galerkin Method II

- How to find u_N in practice?
- Consider a set of basis functions $\{\phi\}_{i=1}^N$ such that $V_N = \text{span}\{\phi\}_{i=1}^N$ and set

$$u_N = \sum_{j=1}^N \xi_j \phi_j$$

In terms of v take the basis functions ϕ_i, $i = 1, \dots, N$

- The weak formulation becomes equivalent to a linear algebraic system

$$A\boldsymbol{\xi} = \mathbf{b},$$

where:
  - $\boldsymbol{\xi} = (\xi_j) \in \mathbb{R}^N$ is the vector of unknown coefficients,
  - $A = \big(a(\phi_i, \phi_j)\big) \in \mathbb{R}^{N\times N}$ is the stiffness matrix
  - $\mathbf{b} = (l(\phi_i)) \in \mathbb{R}^N$ is the load vector
- Approximate solutions with increasing accuracy can be calculated by considering a sequence of nested spaces $V_{N_1} \subseteq V_{N_2} \subseteq \cdots \subseteq V$

The Galerkin Method III

- The Ritz–Galerkin Method can be used when the bilinear form $a(\cdot,\cdot)$ is symmetric, i.e., $a(u,v) = a(v,u)$, $\forall u, v \in V$
- Evidently, the original problem is equivalent to the minimization problem

$$u \in V, \quad E(u) = \inf_{v\in V} E(v),$$

where the energy functional is $E(v) = \frac{1}{2}a(v,v) - l(v)$.

- Note that by considering the directional (Gâteaux) differential of $E(v)$ we obtain

$$E'(u;u') = a(u,u') - l(u') = 0$$

as the necessary condition for optimality

- In the finite–dimensional setting $V_N \subseteq V$ we have

$$u_N \in V_N, \quad E(u_N) = \inf_{v\in V_N} E(v),$$

which can be solved using standard minimization techniques.

- When $a(\cdot,\cdot)$ is symmetric, the Galerkin and Ritz–Galerkin methods are equivalent.

The Galerkin Method IV

- As regards error estimation in the Galerkin method, the key result is Céa's inequality
- Lemma — Assume V is a Hilbert space, $V_N \subseteq V$ is a subspace, $a(\cdot,\cdot)$ is a bounded, V–elliptic bilinear on V, and $l \in V'$. Let $u \in V$ be the solution of the problem

$$u \in V, \quad a(u,v) = l(v), \quad \forall v \in V$$

and $u_N \in V_N$ be the Galerkin approximation defined in

$$u_N \in V_N, \quad a(u_N, v) = l(v), \quad \forall v \in V_N$$

Then there exists a constant C such that

$$\|u - u_N\|_V \le C \inf_{v\in V_N} \|u - v\|_V$$

The Galerkin Method V

- *Proof of Céa's Lemma —*
  - *Subtracting the equation for u_N from that for u and taking $v \in V_N$ we get $a(u - u_N, v) = 0$, $\forall v \in V_N$*
  - *Using this relation together with V–ellipticity and boundedness of $a(\cdot,\cdot)$ we get*

$$C_0\|u-u_N\|_V^2 \le a(u-u_N, u-u_N) = a(u-u_N, u-v) \le M\|u-u_N\|_V\|u-v\|_V$$

Thus $\|u-u_N\|_V \le C\|u-v\|_V$ for any arbitrary $v \in V_N$ ■

- Therefore, to estimate the error of the Galerkin solution, it is sufficient to estimate the approximation error $\inf_{v\in V_N}\|u-v\|_V$
- When $a(\cdot,\cdot)$ is symmetric, it defines an inner product on V whose associated norm $\|v\|_a = \sqrt{a(v,v)}$ is equivalent to $\|v\|_V$. With respect to this new inner product the error of the Galerkin solution $u - u_N$ is orthogonal to V_N

The Galerkin Method VI

- Consequences of Céa's Lemma for convergence analysis
- Proposition — Make the assumptions stated in Céa's Lemma. Assume additionally that $V_{N_1} \subseteq V_{N_2} \subseteq \cdots$ is a sequence of subspaces of V with the property

$$\overline{\bigcup_{i\geq 1} V_{N_i}} = V$$

The the Galerkin method converges, i.e.,

$$\|u - u_{N_i}\|_V \longrightarrow 0, \quad \text{as} \quad i \to \infty$$

- *Proof — By the above (density) assumption we can find a sequence $u_i \in V_{N_i}$, $i \geq 1$ such that*

$$\|u - u_i\|_V \longrightarrow 0, \quad \textit{as} \quad i \to \infty$$

Applying Céa's inequality we have

$$\|u - u_{N_i}\|_V \leq C\|u - u_i\|_V \qquad \blacksquare$$

The Petrov–Galerkin Method I

- Sometimes the solution u and the test function v may belong to different Hilbert spaces, respectively, U and V
- Given $a : U \times V \to \mathbb{R}$ and $l \in V'$, the boundary value problem may have the following weak form

$$u \in U \quad a(u,v) = l(v), \quad \forall v \in V$$

Existence of solutions of this problem is addressed by the generalized Lax–Milgram Lemma

- Such problem can be solved approximately using the Petrov–Galerkin method given by

$$u_N \in U_N \quad a(u_N, v_N) = l(v_N), \quad \forall v_N \in V_N,$$

where $U_N \subseteq U$, $V_N \subseteq V$, and $\dim(U_N) = \dim(V_N) = N$.

- The Petrov–Galerkin method can be proven to be convergent, if certain compatibility conditions for the spaces U_N and V_N are satisfied (the Babušla–Brezzi conditions)