Part III

Review of (Abstract) Approximation Theory

Although this may seem a paradox,
all exact science is dominated by the idea of approximation.
— Bertrand Russell (1872–1970)

Inner Products, Unitary Spaces, Hilbert Spaces

- Consider a real or complex linear space **V**; A SCALAR PRODUCT is real or complex number (x,y) associated with the elements $x,y \in \mathbf{V}$ with the following properties:
  - (x,x) is real, $(x,x) \geq 0$, $(x,x) = 0$ only if $x = 0$,
  - $(x,y) = \overline{(y,x)}$,
  - $(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 (x_1,y) + \alpha_2 (x_2,y)$
- A normed space **V** is said to be UNITARY if its norm and scalar product are connected via the following relation: $\|x\| = (x,x)^{1/2}$
- A complete unitary space **H** is called a HILBERT SPACE

Orthogonality

- Two elements x and y of a Hilbert space **V** are said to be mutually ORTHOGONAL $(x \perp y)$ if $(x,y) = 0$. A countable set of elements $x_1, x_2, \ldots, x_k, \ldots$ is said to be ORTHONORMAL (or to form AN ORTHONORMAL SYSTEMS) if $(x_i, x_j) = \delta_{ij}$
- The following properties hold:
  - $x \perp 0$ for all $x \in \mathbf{V}$
  - $x \perp x$ only if $x = 0$
  - if $x \perp \mathbf{A}$, i.e., $x \perp y$ for all $y \in \mathbf{A} \subseteq \mathbf{V}$, then x is also orthogonal to the linear hull $\mathcal{L}(\mathbf{A})$
  - if $x \perp y_n$ $(n = 1, 2, \ldots)$ and $y_n \to y$, then $x \perp y$
  - if **A** is dense in **V** and $x \perp \mathbf{A}$, then $x = 0$
- SCHMIDT ORTHOGONALIZATION — Let **A** be a set of countably many linearly independent elements $x_1, x_2, \ldots, x_k, \ldots$ of a Hilbert space **H**. Then there is an orthonormal system $\mathbf{F} = \{e_i \in \mathbf{V} : (e_i, e_j) = \delta_{ij}\}$, such that the linear hulls of $\mathbf{A}_k = \{x_j : j = 1, \ldots, k\}$ and $\mathbf{F}_k = \{e_j : j = 1, \ldots, k\}$ are the same for all k.

Approximation in Hilbert Spaces (I)

- Let $\{e_1, e_2, \ldots\}$ be an orthonormal system in a Hilbert space **H** and let $\mathbf{H}_k$ be the linear hull of $\{e_1, \ldots, e_k\}$. Then for every $x \in \mathbf{H}$ the element $a = \sum_{j=1}^{k} (x, e_j)\, e_j \in \mathbf{H}_k$ has the property that $\|x - a\| \leq \|x - y\|$ for all $y \in \mathbf{H}_k$. The numbers (x, e_j) are called THE FOURIER COEFFICIENTS relative to the orthonormal system $\{e_1, e_2, \ldots\}$. Furthermore, from $\|x - a\|^2 \geq 0$ follows the BESSEL INEQUALITY :

$$\sum_{j=1}^{k} |(x, e_j)|^2 \leq (x, x)$$

- If **A** is a given set in a Hilbert space **H**, then

$$\mathbf{A}^{\perp} = \{x : (x, a) = 0 \text{ for all } a \in \mathbf{A}\}$$

is a closed linear subspace of **H**. It is, therefore, itself a Hilbert space and is called THE ORTHOGONAL COMPLEMENT OF **A**

Approximation in Hilbert spaces (II)

- If $\mathbf{H}_1$ is a closed linear subspace of a Hilbert space $\mathbf{H}$ and $\mathbf{H}_2$ is its orthogonal complement, then every $x \in \mathbf{H}$ can be uniquely represented in the form

$$x = x_1 + x_2, \quad (x_1 \in \mathbf{H}_1, x_2 \in \mathbf{H}_2)$$

One writes $\mathbf{H} = \mathbf{H}_1 \oplus \mathbf{H}_2$ and calls $\mathbf{H}$ an orthogonal sum of $\mathbf{H}_1$ and $\mathbf{H}_2$. Since

$$\|x - x_1\| = \rho(x, \mathbf{H}_1) = \inf_{y_1 \in \mathbf{H}_1} \{\|x - y_1\|\},$$

$$\|x - x_2\| = \rho(x, \mathbf{H}_2) = \inf_{y_2 \in \mathbf{H}_2} \{\|x - y_2\|\},$$

one calls x_1 and x_2 the ORTHOGONAL PROJECTIONS of x on $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively.

Approximation in Hilbert spaces (III)

- Let $\{e_1, e_2, \dots\}$ be a countable orthonormal system in a Hilbert space $\mathbf{H}$. By Bessel inequality, the series $\sum_{j=1}^{\infty} (x, e_j)\, e_j = \lim_{n\to\infty} \sum_{j=1}^{n} (x, e_j)\, e_j$ defines an element of $\mathbf{H}$ for every $x \in \mathbf{H}$. This is called THE FOURIER SERIES OF x
- The partial sum $s_n = \sum_{j=1}^{n} (x, e_j)\, e_j$ is the orthogonal projection of x on the subspace $\mathbf{H}_n = \mathcal{L}(\{e_1, \dots, e_n\})$. One has $\|s_n\|^2 = \sum_{j=1}^{n} |(x, e_j)|^2$
- If the system $\{e_1, \dots, e_k, \dots\}$ is complete in $\mathbf{H}$, i.e., $\overline{\mathcal{L}(\{e_1, \dots, e_k, \dots\})} = \mathbf{H}$, then the Fourier series for any $x \in \mathbf{H}$ converges to x
- An orthonormal system is said to be CLOSED if THE PARCEVAL EQUATION

$$\sum_{j=1}^{\infty} |(x, e_j)|^2 = \|x\|^2$$

holds for every $x \in \mathbf{H}$. An orthonormal system is closed IFF it is complete.
- An orthonormal system in a separable Hilbert space is at most countable

Approximation in Hilbert spaces (IV)

- Statement of a GENERAL APPROXIMATION PROBLEM IN A HILBERT SPACE $\mathbf{H}$ — consider a fixed element $f \in \mathbf{H}$ and $\mathbf{G_n} \subseteq \mathbf{H}$ which is a finite–dimensional subspace of $\mathbf{H}$ (with the same norm). Want to find an element $\hat{g} \in \mathbf{G_n}$ such that

$$D(f, \mathbf{G_n}, \|\cdot\|) \triangleq \inf_{\mathbf{g} \in \mathbf{G_n}} \{\|\mathbf{f} - \mathbf{g}\|\} = \|\mathbf{f} - \hat{\mathbf{g}}\|$$

The element $\hat{g}$ is called THE BEST APPROXIMATION and the number $D(f, \mathbf{G_n}, \|\cdot\|)$ is called THE DEFECT .
- Issues:
  - Does the best approximation $\hat{g}$ exist?
  - Can $\hat{g}$ be uniquely determined?
  - How can $\hat{g}$ be computed?

Approximation in Hilbert spaces (V)

- The approximation problem in a Hilbert space $\mathbf{H}$ has a unique solution $\hat{g}$ for which $(\hat{g} - f, h) = 0$ holds for all $h \in \mathbf{G_n}$. If $\{e_1, \dots, e_n\}$ is a basis of $\mathbf{G_n}$, then

$$\hat{g} = \sum_{j=1}^{n} c_j^{(n)} e_j$$

with

$$\sum_{j=1}^{n} c_j^{(n)} (e_j, e_k) = (f, e_k), \quad j = 1, \dots, n \qquad (\bigstar)$$

and

$$\|f - \hat{g}\|^2 = (f - \hat{g}, f - \hat{g}) = \|f\|^2 - \sum_{j=1}^{n} c_j^{(n)} (e_j, f)$$

- Thus, the Fourier coefficients $c_j^{(n)}$ $j = 1, \dots, n$ can be calculated by solving an algebraic system $(\bigstar)$ with the Hermitian, positive–definite matrix $A_{jk} = (e_j, e_k)$ (the so called GRAM MATRIX).
- If the basis $\{e_1, \dots, e_n\}$ is orthogonal, the system becomes decoupled and the Fourier coefficients can be calculated simply as $c_k^{(n)} = (f, e_k)$

APPROXIMATION IN HILBERT SPACES (VI)
RATES OF CONVERGENCE

- Assume that $c_j,\ j = 1, 2, \ldots$ are the Fourier coefficients related to an approximation of some function $f = \sum_{j=1}^{n} c_j e_j$
- The RATE OF CONVERGENCE of this approximation is:
  - ALGEBRAIC with order k if for $j >> 1$

$$\lim_{j\to\infty} |c_j| j^k < \infty, \quad \text{or, equivalently,} \quad |c_j| \sim \mathcal{O}[j^{-k}]$$

  - EXPONENTIAL OR SPECTRAL with index r if for ANY $j > 0$

$$\lim_{j\to\infty} |c_j| j^k < \infty, \quad \text{or, equivalently,} \quad |c_j| \sim \mathcal{O}[\exp(-q j^r)], \quad r, q \in \mathbb{R}^+$$

    spectral convergence can be:
    * SUBGEOMETRIC when $r < 1$,
    * GEOMETRIC when $r = 1$, and
    * SUPERGEOMETRIC otherwise