# PART IV

## Spectral Methods

- ADDITIONAL REFERENCES:
  - R. Peyret, *Spectral methods for incompressible viscous flow*, Springer (2002),
  - B. Mercier, *An introduction to the numerical analysis of spectral methods*, Springer (1989),
  - C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer (1988).

## METHOD OF WEIGHTED RESIDUALS (I)

- SPECTRAL METHODS belong to the broader category of WEIGHTED RESIDUAL METHODS , for which approximations are defined in terms of series expansions, such that a measure of the error knows as the RESIDUAL is set to be zero in some approximate sense
- In general, an approximation $u_N(x)$ to $u(x)$ is constructed using a set of basis functions $\varphi_k(x)$, $k = 0,\dots,N$ (note that $\varphi_k(x)$ need not be ORTHOGONAL )

$$u_N(x) \triangleq \sum_{k\in I_N} \hat{u}_k \varphi_k(x), \quad a \le x \le b, \quad I_N = \{1,\dots,N\}$$

- Residual for two central problems:
  - APPROXIMATION of a function $u$:

$$R_N(x) = u - u_N$$

  - APPROXIMATE SOLUTION of a (differential) equation $\mathcal{L}u - f = 0$:

$$R_N(x) = \mathcal{L}u_N - f$$

## METHOD OF WEIGHTED RESIDUALS (II)

- In general, the residual $R_N$ in canceled in the following sense:

$$(R_N,\psi_i)_{w_*} = \int_a^b w_* R_N \bar{\psi}_i \, dx = 0, \quad i \in I_N,$$

  where $\psi_i(x)$, $i \in I_N$ are the TRIAL (TEST) FUNCTIONS and $w : [a,b] \to \mathbb{R}^+$ are the WEIGHTS
- Spectral Method is obtained by:
  - selecting the BASIS FUNCTIONS $\varphi_k$ to form an ORTHOGONAL system under the weight $w$:

$$(\varphi_i,\varphi_k)_w = \delta_{ik}, \quad i,k \in I_N \quad \text{and}$$

  - selecting the trial functions to coincide with the basis functions:

$$\psi_k = \varphi_k, \quad k \in I_N$$

    with the weights $w_* = w$ ( SPECTRAL GALERKIN APPROACH ), or
  - selecting the trial functions as

$$\psi_k = \delta(x - x_k), \quad x_k \in (a,b),$$

    where $x_k$ are chosen in a non–arbitrary manner, and the weights are $w_* = 1$ ( COLLOCATION, "PSEUDO–SPECTRAL" APPROACH )

## METHOD OF WEIGHTED RESIDUALS (III)

- Note that the residual $R_N$ vanishes
  - in the mean sense specified by the weight $w$ in the Galerkin approach
  - pointwise at the points $x_k$ in the collocation approach

# Approximation of Functions (I) — Galerkin Method

- Assume that the basis functions $\{\varphi_k\}_{k=1}^N$ form an orthogonal set
- Define the residual

$$R_N(x) = u - u_N = u - \sum_{k=0}^{N} \hat{u}_k \varphi_k$$

- Cancellation of the residual in the mean sense (with the weight $w$)

$$(R_N, \varphi_i)_w = \int_a^b \left( u - \sum_{k=0}^{N} \hat{u}_k \varphi_k \right) \bar{\varphi}_i \, w \, dx = 0, \quad i = 0, \dots, N$$

  $\bar{(\cdot)}$ denotes complex conjugation (cf. definition of the inner product)

- Orthogonality of the basis / trial functions thus allows us to determine the coefficients $\hat{u}_k$ by evaluating the expressions

$$\hat{u}_k = \int_a^b u \bar{\varphi}_k \, w \, dx, \quad k = 0, \dots, N$$

- Note that, for this problem, the Galerkin approach is equivalent to the LEAST SQUARES METHOD .

# Approximation of Functions (II) — Collocation Method

- Define the residual

$$R_N(x) = u - u_N = u - \sum_{k=0}^{N} \hat{u}_k \varphi_k$$

- POINTWISE cancellation of the residual

$$\sum_{k=0}^{N} \hat{u}_k \varphi_k(x_i) = u(x_i), \quad i = 0, \dots, N$$

  Determination of the coefficients $\hat{u}_k$ thus requires solution of an algebraic system. Existence and uniqueness of solutions requires that $\det\{\varphi_k(x_i)\} \neq 0$ (condition on the choice of the collocation points $x_j$ and the basis functions $\varphi_k$)

- For certain basis pairs of basis functions $\varphi_k$ and collocation points $x_j$ the above system can be easily inverted and therefore determination of $\hat{u}_k$ may be reduced to evaluation of simple expressions
- For this problem, the collocation method thus coincides with an INTERPOLATION TECHNIQUE based on the set of points $\{x_j\}$

# Approximation of PDEs (I) — Galerkin Method

- Consider a generic PDE problem

$$\begin{cases} \mathcal{L}u - f = 0 & a < x < b \\ \mathcal{B}_- u = g_- & x = a \\ \mathcal{B}_+ u = g_+ & x = b, \end{cases}$$

  where $\mathcal{L}$ is a linear, second–order differential operator, and $\mathcal{B}_-$ and $\mathcal{B}_+$ represent appropriate boundary conditions (Dirichlet, Neumann, or Robin)

- Reduce the problem to an equivalent HOMOGENEOUS formulation via a “lifting” technique, i.e., substitute $u = \tilde{u} + v$ , where $\tilde{u}$ is an arbitrary function satisfying the boundary conditions above and the new (homogeneous) problem for $v$ is

$$\begin{cases} \mathcal{L}v - h = 0 & a < x < b \\ \mathcal{B}_- v = 0 & x = a \\ \mathcal{B}_+ v = 0 & x = b, \end{cases}$$

  where $h = f - \mathcal{L}\tilde{u}$

- The reason for this transformation is that the basis functions $\varphi_k$ (usually) satisfy homogeneous boundary conditions.

# Approximation of PDEs (II) — Galerkin Method

- The residual

$$R_N(x) = \mathcal{L}v_N - h, \quad \text{where} \quad v_N = \sum_{k=0}^{N} \hat{v}_k \varphi_k(x)$$

  satisfies (“by construction”) the boundary conditions

- Cancellation of the residual in the mean (cf. THE WEAK FORMULATION )

$$(R_N, \varphi_i)_w = (\mathcal{L}v_N - h, \varphi_i)_w, \quad i = 0, \dots, N$$

  Thus

$$\sum_{k=0}^{N} \hat{v}_k (\mathcal{L}\varphi_k, \varphi_i)_w = (h, \varphi_i)_w, \quad i = 0, \dots, N,$$

  where the scalar product $(\mathcal{L}\varphi_k, \varphi_i)_w$ can be accurately evaluated using properties of the basis functions $\varphi_i$ and $(h, \varphi_i)_w = \hat{h}_i$

- An $(N+1) \times (N+1)$ algebraic system is obtained with the matrix determined by
  - the properties of the basis functions $\{\varphi_k\}_{k=1}^N$
  - the properties of the operator $\mathcal{L}$

## Approximation of PDEs (III) — Collocation Method

- The residual (corresponding to the original inhomogeneous problem)

$$R_N(x) = \mathcal{L}u_N - f, \quad \text{where} \quad u_N = \sum_{k=0}^{N} \hat{u}_k \varphi_k(x)$$

- Pointwise cancellation of the residual, including the boundary nodes:

$$\begin{cases} \mathcal{L}u_N(x_i) = f(x_i) & i = 1,\dots,N-1 \\ \mathcal{B}_- u_N(x_0) = g_- \\ \mathcal{B}_+ u_N(x_N) = g_+, \end{cases}$$

  This results in an $(N+1)\times(N+1)$ algebraic system. Note that depending on the properties of the basis $\{\varphi_0,\dots,\varphi_N\}$, this system may be singular.

- Sometimes an alternative formulation is useful, where the nodal values $u_N(x_j)$ $j = 0,\dots,N$, rather than the expansion coefficients $\hat{u}_k$, $k = 0,\dots,N$ are unknown. The advantage is a convenient form of the expression for the derivative

$$u_N^{(p)}(x_i) = \sum_{j=0}^{N} d_{ij}^{(p)} u_N(x_j),$$

  where $d^{(p)}$ is a $p$–th order differentiation matrix .

## Orthonormal systems (I) — Construction

- *THEOREM* — Let $\mathbf{H}$ be a separable Hilbert space and $\mathcal{T}$ a compact Hermitian operator. Then, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{W_n\}_{n\in\mathbb{N}}$ such that
  1. $\lambda_n \in \mathbb{R}$,
  2. the family $\{W_n\}_{n\in\mathbb{N}}$ forms a complete basis in $\mathbf{H}$
  3. $\mathcal{T}W_n = \lambda_n W_n$ for all $n \in \mathbb{N}$
- Systems of orthogonal functions are therefore related to spectra of certain operators, hence the name spectral methods

## Orthonormal systems (II) — Example # 1

- Let $\mathcal{T}: L_2(0,\pi) \to L_2(0,\pi)$ be defined for all $f \in L_2(0,\pi)$ by $\mathcal{T}f = u$, where $u$ is the solution of the Dirichlet problem

$$\begin{cases} -u'' = f \\ u(0) = u(\pi) = 0 \end{cases}$$

  Compactness of $\mathcal{T}$ follows from the Lax–Milgram lemma and compact embeddedness of $H^1(0,\pi)$ in $L_2(0,\pi)$.

- Eigenvalues and eigenvectors

$$\lambda_k = \frac{1}{k^2} \quad \text{and} \quad W_k = \sqrt{2}\sin(kx) \quad \text{for} \quad k \ge 1$$

- Thus, each function $u \in L_2(0,\pi)$ can be represented as

$$u(x) = \sqrt{2}\sum_{k\ge 1} \hat{u}_k W_k(x),$$

  where $\hat{u}_k = (u, W_k)_{L_2} = \frac{\sqrt{2}}{\pi}\int_0^{\pi} u(x)\sin(kx)\,dx$ .

- Uniform (pointwise) convergence is not guaranteed (only in $L_2$ sense)!

## Orthonormal systems (III) — Example # 2

- Let $\mathcal{T}: L_2(0,\pi) \to L_2(0,\pi)$ be defined for all $f \in L_2(0,\pi)$ by $\mathcal{T}f = u$, where $u$ is the solution of the Neumann problem

$$\begin{cases} -u'' + u = f \\ u'(0) = u'(\pi) = 0 \end{cases}$$

  Compactness of $\mathcal{T}$ follows from the Lax–Milgram lemma and compact embeddedness of $H^1(0,\pi)$ in $L_2(0,\pi)$.

- Eigenvalues and eigenvectors

$$\lambda_k = \frac{1}{1+k^2} \quad \text{and} \quad W_0(x) = 1, \; W_k = \sqrt{2}\cos(kx) \quad \text{for} \quad k > 1$$

- Thus, each function $u \in L_2(0,\pi)$ can be represented as

$$u(x) = \sqrt{2}\sum_{k\ge 0} \hat{u}_k W_k(x),$$

  where $\hat{u}_k = (u, W_k)_{L_2} = \frac{\sqrt{2}}{\pi}\int_0^{\pi} u(x)\cos(kx)\,dx$ .

- Uniform (pointwise) convergence is not guaranteed (only in $L_2$ sense)!

# Orthonormal systems (IV) — Example # 3

- Expansion in sine series good for functions vanishing on the boundaries
- Expansion in cosine series good for functions with first derivatives vanishing on the boundaries
- Combining sine and cosine expansions we obtain the Fourier series expansion with the basis functions (in $L_2(-\pi,\pi)$)

  $$W_k(x)=e^{ikx},\quad \text{for } k\geq 0$$

  $W_k$ form a Hilbert basis with better properties then sine or cosine series alone.
- Fourier series vs. Fourier transform —
  - Fourier Transform : $\mathcal{F}_1 : L_2(\mathbb{R})\to L_2(\mathbb{R})$,

    $$\mathcal{F}_1[u](k)=\int_{-\infty}^{\infty} e^{-ikx}u(x)\,dx,\quad k\in\mathbb{R}$$
  - Fourier Series : $\mathcal{F}_2 : L_2(0,2\pi)\to l_2$, (i.e., bounded to discrete)

    $$\hat{u}_k=\mathcal{F}_2[u](k)=\int_0^{2\pi} e^{-ikx}u(x)\,dx,\quad k=0,1,2,\dots$$

# Orthonormal systems (V) — Polynomial Approximation

- Weierstrass Approximation Theorem — To any function $f(x)$ that is continuous in $[a,b]$ and to any real number $\varepsilon>0$ there corresponds a polynomial $P(x)$ such that $\|P(x)-f(x)\|_{C(a,b)}<\varepsilon$, i.e. the set of polynomials is dense in the Banach space $C(a,b)$

  ($C(a,b)$ is the Banach space with the norm $\|f\|_{C(a,b)}=\max_{x\in[a,b]}|f(x)|$
- Thus the power functions $x^k$, $k=0,1,\dots$ represent a natural basis in $C(a,b)$
- Question — Is this set of basis functions useful?

  No! — see below

# Orthonormal systems (VI) — Example

- Find the polynomial $\bar{P}_N$ (of order $N$) that best approximates a function $f\in L_2(a,b)$ [note that we will need the structure of a Hilbert space, hence we go to $L_2(a,b)$, but $C(a,b)\subset L_2(a,b)$], i.e.

  $$\int_a^b[f(x)-\bar{P}_N(x)]^2\,dx\leq\int_a^b[f(x)-P_N(x)]^2\,dx$$

  where

  $$\bar{P}_N(x)=\bar{a}_0+\bar{a}_1x+\bar{a}_2x^2+\cdots+\bar{a}_Nx^N$$
- Using the formula $\sum_{j=0}^N\bar{a}_j(e_j,e_k)=(f,e_k)$, $j=0,\dots,N$, where $e_k=x^k$

  $$\sum_{k=0}^N\bar{a}_k\int_a^b x^{k+j}\,dx=\int_a^b x^j f(x)\,dx$$

  $$\sum_{k=0}^N\bar{a}_k\frac{b^{k+j+1}-a^{k+j+1}}{k+j+1}=\int_a^b x^j f(x)\,dx$$
- The resulting algebraic problem is extremely ill–conditioned , e.g. for $a=0$ and $b=1$

  $$[A]_{kj}=\frac{1}{k+j+1}$$

# Orthonormal systems (VII) — Polynomial Approximation

- Much better behaved approximation problems are obtained with the use of orthogonal basis functions
- Such systems of orthogonal basis functions are derived by applying the Schmidt orthogonalization procedure to the system $\{1,x,\dots,x^N\}$
- Various families of orthogonal polynomials are obtained depending on the choice of:
  - the domain $[a,b]$ over which the polynomials are defined, and
  - the weight $w$ characterizing the inner product $(\cdot,\cdot)_w$ used for orthogonalization

## ORTHONORMAL SYSTEMS (VIII) — ORTHOGONAL POLYNOMIALS

- Polynomials defined on the interval $[-1,1]$
  - LEGENDRE POLYNOMIALS $(w=1)$

$$P_k(x)=\sqrt{\frac{2k+1}{2}}\frac{1}{2^k k!}\frac{d^k}{dx^k}(x^2-1)^k,\;\; k=0,1,2,\dots$$

  - JACOBI POLYNOMIALS $(w=(1-x)^{\alpha}(1+x)^{\beta})$

$$J_k^{(\alpha,\beta)}(x)=C_k(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^k}{dx^k}[(1-x)^{\alpha+k}(1+x)^{\beta+k}]\;\; k=0,1,2,\dots,$$

  where $C_k$ is a very complicated constant
  - CHEBYSHEV POLYNOMIALS $(w=\frac{1}{\sqrt{1-x^2}})$

$$T_n(x)=\cos(k\arccos(x)),\;\; k=0,1,2,\dots,$$

  Note that Chebyshev polynomials are obtained from Jacobi polynomials for $\alpha=\beta=-1/2$

## ORTHONORMAL SYSTEMS (IX) — ORTHOGONAL POLYNOMIALS

- Polynomials defined on the PERIODIC interval $[-\pi,\pi]$
  TRIGONOMETRIC POLYNOMIALS $(w=1)$

$$S_k(x)=e^{ikx}\;\; k=0,1,2,\dots$$

- Polynomials defined on the interval $[0,+\infty]$
  LAGUERRE POLYNOMIALS $(w=e^{-x})$

$$L_k(x)=\frac{1}{k!}e^x\frac{d^k}{dx^k}(e^{-x}x^k),\;\; k=0,1,2,\dots$$

- Polynomials defined on the interval $[-\infty,+\infty]$
  HERMITE POLYNOMIALS $(w=1)$

$$H_k(x)=\frac{(-1)^k}{(2^k k!\sqrt{\pi})^{1/2}}e^{x^2}\frac{d^k}{dx^k}e^{-x^2},\;\; k=0,1,2,\dots$$

## ORTHONORMAL SYSTEMS (X) — ORTHOGONAL POLYNOMIALS

- What is the relationship between ORTHOGONAL POLYNOMIALS and eigenfunctions of a COMPACT HERMITIAN OPERATOR (cf. Theorem on page 75)?
- Each of the aforementioned families of ORTHOGONAL POLYNOMIALS forms the set of eigenvectors for the following STURM–LIOUVILLE PROBLEM

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+[q(x)+\lambda r(x)]y=0$$
$$a_1y(a)+a_2y'(a)=0$$
$$b_1y(b)+b_2y'(b)=0$$

  for appropriately selected domain $[a,b]$ and coefficients $p$, $q$, $r$, $a_1$, $a_2$, $b_1$, $b_2$.

## FOURIER SERIES (I) — CALCULATION OF FOURIER COEFFICIENTS

- TRUNCATED FOURIER SERIES:

$$u_N(x)=\sum_{k=-N}^{N}\hat{u}_k e^{ikx}$$

- The series involves $2N+1$ complex coefficients of the form (weight $w\equiv 1$):

$$\hat{u}_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}ue^{-ikx}\,dx,\;\; k=-N,\dots,N$$

- The expansion is redundant for real–valued $u$ — the property of CONJUGATE SYMMETRY $\hat{u}_{-k}=\overline{\hat{u}_k}$, which reduces the number of complex coefficients to $N+1$; furthermore, $\Im(\hat{u}_0)\equiv 0$ for real $u$, thus one has $2N+1$ REAL coefficients; in the real case one can work with positive frequencies only!
- Equivalent real representation:

$$u_N(x)=a_0+\sum_{k=1}^{N}[a_k\cos(kx)+b_k\sin(kx)],$$

  where $a_0=\hat{u}_0$, $a_k=2\Re(\hat{u}_k)$ and $b_k=2\Im(\hat{u}_k)$.

# Fourier series (II) — Uniform convergence

- Consider a function $u$ that is continuous, periodic (with the period $2\pi$) and differentiable; note the following two facts:
  - The Fourier coefficients are always less than the average of $u$
$$|\hat{u}_k| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{ikx}\,dx\right| \leq M(u) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi}|u(x)|\,dx$$
  - If $v = \frac{d^\alpha u}{dx^\alpha} = u^{(\alpha)}$, then $\hat{u}_k = \frac{\hat{v}_k}{(ik)^\alpha}$
- Then, using integration by parts, we have
$$\hat{u}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx = \frac{1}{2\pi}\left[u(x)\frac{e^{-ikx}}{-ik}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} u'(x)\frac{e^{-ikx}}{-ik}\,dx$$
- Repeating integration by parts $p$ times
$$\hat{u}_k = (-1)^p \frac{1}{2\pi}\int_{-\pi}^{\pi} u^{(p)}(x)\frac{e^{-ikx}}{(-ik)^p}\,dx \implies |\hat{u}_k| \leq \frac{M(u^{(p)})}{|k|^p}$$

  Therefore, the more regular is the function $u$, the more rapidly its Fourier coefficients tend to zero as $|n| \to \infty$

# Fourier series (III) — Uniform convergence

- We have
$$|\hat{u}_k| \leq \frac{M(u'')}{|k|^2} \implies \sum_{k\in\mathbb{Z}} |\hat{u}_k e^{ikx}| \leq \hat{u}_0 + \sum_{n\neq 0}\frac{M(u'')}{n^2}$$
  The latter series converges absolutely
- Thus, if $u$ is twice continuously differentiable and its first derivative is continuous and periodic with period $2\pi$, then its Fourier series $u_N = P_N u$ converges uniformly to $u$ for $|N| \to \infty$
- Spectral convergence – if $\phi \in C_p^\infty(-\pi,\pi)$, then for all $\alpha > 0$ there exists a positive constant $C_\alpha$ such that $|\hat{\phi}_k| \leq \frac{C_\alpha}{|n|^\alpha}$, i.e., for a function with an infinite number of smooth derivatives, the Fourier coefficients vanish faster than algebraically

# Fourier series (IV) — Rates of Convergence

- Rate of decay of Fourier transform of a function $f:\mathbb{R}\to\mathbb{R}$ is determined by its smoothness ; functions defined on a bounded (periodic) domain are a special case
- *THEOREM [a collection of several related results, see also Trefethen (2000)]* — Let $u \in L_2(\mathbb{R})$ have Fourier transform $\hat{u}$.
  - If $u$ has $p-1$ continuous derivatives in $L_2(\mathbb{R})$ for some $p \geq 0$ and a $p$–th derivative of bounded variation, then $\hat{u}(k) = O(|k|^{-p-1})$ as $|k| \to \infty$,
  - If $u$ has infinitely many continuous derivatives in $L_2(\mathbb{R})$, then $\hat{u}(k) = O(|k|^{-m})$ as $|k| \to \infty$ for every $m \geq 0$ (the converse also holds)
  - If there exist $a, c > 0$ such that $u$ can be extended to an analytic function in the complex strip $|\Im(z)| < a$ with $\|u(\cdot + iy)\| \leq c$ uniformly for all $y \in (-a,a)$, where $\|u(\cdot + iy)\|$ is the $L_2$ norm along the horizontal line $\Im(z) = y$, then $u_a \in L_2(\mathbb{R})$, where $u_a(k) = e^{a|k|}\hat{u}(k)$ (the converse also holds)
  - If $u$ can be extended to an entire function (i.e., analytic throughout the complex plane) and there exists $a > 0$ such that $|u(z)| = o(e^{a|z|})$ as $|z| \to \infty$ for all complex values $z \in \mathbb{C}$, the $\hat{u}$ has compact support contained in $[-a,a]$; that is $\hat{u}(k) = 0$ for all $|k| > a$ (the converse also holds)

# Fourier series (V) — Radii of Convergence

- Darboux's Principle [see Boyd (2001)] — for all types of spectral expansions (and for ordinary power series), both the domain of convergence in the complex plane and the rate of convergence are controlled by the location and strength of the gravest singularity in the complex plane ("singularities" in this context denote poles, fractional powers, logarithms and discontinuities of $f(z)$ or its derivatives)
- Thus, given a function $f:[0,2\pi]\to\mathbb{R}$, the rate of convergence of its Fourier series is determined by the properties of its complex extension $F:\mathbb{C}\to\mathbb{C}$!!!
- Shapes of regions of convergence:
  - Taylor series — circular disk extending up to the nearest singularity
  - Fourier (and Hermite) series — horizontal strip extending vertically up to the nearest singularity
  - Chebyshev series — ellipse with foci at $x = \pm 1$ and extending up to the nearest singularity

## FOURIER SERIES (VI) — PERIODIC SOBOLEV SPACES

- Let $H^r_p(I)$ be a PERIODIC SOBOLEV SPACE , i.e.,

$$H^r_p(I) = \{u : u^{(\alpha)} \in L_2(I), \alpha = 0, \dots, r\},$$

where $I = (-\pi,\pi)$ is a periodic interval. The space $C^\infty_p(I)$ is dense in $H^r_p(I)$

- The following two norms can be shown to be EQUIVALENT in $H^r_p$:

$$\|u\|_r = \left[\sum_{k\in\mathbb{Z}} (1+k^2)^r |\hat{u}_k|^2\right]^{1/2}$$

$$|||u|||_r = \left[\sum_{\alpha=0}^{r} C^\alpha_r \|u^{(\alpha)}\|^2\right]^{1/2}$$

Note that the first definition is naturally generalized for the case when $r$ is non–integer!

- The PROJECTION OPERATOR $P_N$ commutes with the derivative in the distribution sense:

$$(P_N u)^{(\alpha)} = \sum_{|k|\le N} (ik)^\alpha \hat{u}_k W_k = P_N u^{(\alpha)}$$

## FOURIER SERIES (VII) — APPROXIMATION ERROR ESTIMATES IN $H^s_p(I)$

- Let $r,s \in \mathbb{R}$ with $0 \le s \le r$; then we have:

$$\|u - P_N u\|_s \le (1+N^2)^{\frac{s-r}{2}} \|u\|_r, \ \text{ for } \ u \in H^r_p(I)$$

Proof:

$$\|u-P_N u\|^2_s = \sum_{|k|>N} (1+k^2)^{s-r+r} |\hat{u}_k|^2 \le (1+N^2)^{s-r} \sum_{|k|>N} (1+k^2)^r |\hat{u}_k|^2$$
$$\le (1+N^2)^{s-r} \|u\|^2_r$$

- Thus, accuracy of the approximation $P_N u$ is better when $u$ is SMOOTHER ; more precisely, for $u \in H^r_p(I)$, the $L_2$ leading order error is $O(N^{-r})$ which improves when $r$ increases.

## FOURIER SERIES (VIII) — APPROXIMATION ERROR ESTIMATES IN $L_\infty(I)$

- First, a useful lemma (SOBOLEV INEQUALITY) — let $u \in H^1_p(I)$, then there exists a constant $C$ such that

$$\|u\|^2_{L_\infty(I)} \le C\|u\|_0 \|u\|_1$$

Proof: Suppose $u \in C^\infty_p(I)$; note the following facts

  - $\hat{u}_0$ is the average of $u$
  - From the mean value theorem: $\exists x_0 \in I$ such that $\hat{u}_0 = u(x_0)$

Let $v(x) = u(x) - \hat{u}_0$, then

$$\frac{1}{2}|v(x)|^2 = \int_{x_0}^{x} v(y)v'(y)\,dy \le \left(\int_{x_0}^{x} |v(y)|^2\,dy\right)^{1/2} \left(\int_{x_0}^{x} |v'(y)|^2\,dy\right)^{1/2} \le 2\pi\|v\|\,\|v'\|$$

$$|u(x)| \le |\hat{u}_0| + |v(x)| \le |\hat{u}_0| + 2\pi^{1/2}\|v\|^{1/2}\,\|v'\|^{1/2} \le C\|u\|_0^{1/2}\,\|u\|_1^{1/2},$$

since $v' = u'$, $\|v\| \le \|u\|$ and $|\hat{u}_0| \le \|u\|$.
As $C^\infty_p(I)$ is dense in $H^1_p(I)$, the inequality also holds for any $u \in H^1_p(I)$.

## FOURIER SERIES (IX) — APPROXIMATION ERROR ESTIMATES IN $L_\infty(I)$

- An estimate in the norm $L_\infty(I)$ follows immediately from the previous lemma and estimates in the $H^s_p(I)$ norm

$$\|u - P_N u\|^2_{L_\infty(I)} \le C(1+N^2)^{-\frac{r}{2}}(1+N^2)^{\frac{1-r}{2}},$$

where $u \in H^r_p(I)$

- Thus for $r \ge 1$

$$\|u - P_N u\|^2_{L_\infty(I)} = O(N^{\frac{1}{2}-r})$$

- UNIFORM CONVERGENCE for all $u \in H^1_p(I)$
(Note that $u$ need only to be CONTINUOUS , therefore this result is stronger than the one given on page 87)

## Fourier series (X) — Spectral Differentiation

- Assume we have a truncated Fourier series of $u(x)$

$$u_N(x) = P_N u(x) = \sum_{k=-N}^{N} \hat{u}_k e^{ikx}$$

- The Fourier series of the $p$–th derivative of $u(x)$ is

$$u_N^{(p)}(x) = P_N u^{(p)} = \sum_{k=-N}^{N} (ik)^p \hat{u}_k e^{ikx} = \sum_{k=-N}^{N} \hat{u}_k^{(p)} e^{ikx}$$

- Thus, using the vectors $\hat{U} = [\hat{u}_{-N}, \dots, \hat{u}_N]^T$ and $\hat{U}^{(p)} = [\hat{u}_{-N}^{(p)}, \dots, \hat{u}_N^{(p)}]^T$, one can introduce the Spectral Differentiation Matrix $\mathbb{D}^{(p)}$ defined in Fourier space as $\hat{U}^{(p)} = \hat{\mathbb{D}}^{(p)} \hat{U}$ , where

$$\hat{\mathbb{D}}^{(p)} = i^p \begin{bmatrix} -N^p & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & N^p \end{bmatrix}$$

## Fourier series (XI) — Spectral Differentiation

- Properties of the spectral differentiation matrix in Fourier representation
  - $\mathbb{D}^{(p)}$ is DIAGONAL
  - $\mathbb{D}^{(p)}$ is SINGULAR (diagonal matrix with a zero eigenvalue)
  - after desingularization the 2–norm condition number of $\mathbb{D}^{(p)}$ grows in proportion to $N^p$ (since the matrix is diagonal, this is not an issue)
- Question — how to derive the corresponding spectral differentiation matrix in REAL REPRESENTATION ?

Will see shortly ...

## Spectral Galerkin Method — Numerical Quadratures (I)

- We need to evaluate the expansion (Fourier) coefficients

$$\hat{u}_k = (u, \phi_k)_w = \int_a^b w(x) u(x) \phi_k(x)\, dx, \quad k = 0, \dots, N$$

- Quadrature is a method to evaluate such integrals approximately.
- Gaussian Quadrature seeks to obtain the best numerical estimate of an integral $\int_a^b w(x)\, f(x)\, dx$ by picking OPTIMAL POINTS $x_i$, $i = 1, \dots, N$ at which to evaluate the function $f(x)$.
- The Gauss–Jacobi Integration Theorem — If the $(N+1)$ interpolation points $\{x_i\}_{i=0}^N$ are chosen to be the zeros of $P_{N+1}(x)$, where $P_{N+1}(x)$ is the polynomial of degree $(N+1)$ of the set of polynomials which are orthogonal on $[a,b]$ with respect to the weight function $w(x)$, then the quadrature formula

$$\int_a^b w(x) f(x)\, dx = \sum_{i=0}^{N} w_i f(x_i)$$

is EXACT for all $f(x)$ which are polynomials of at most degree $(2N+1)$

## Spectral Galerkin Method — Numerical Quadratures (II)

- Definition — Let $K$ be a non-empty, Lipschitz, compact subset of $\mathbb{R}^d$. Let $l_q \geq 1$ be an integer. A quadrature on $K$ with $l_q$ points consists of:
  - A set of $l_q$ real numbers $\{\omega_1, \dots, \omega_{l_q}\}$ called QUADRATURE WEIGHTS
  - A set of $l_q$ points $\{\xi_1, \dots, \xi_{l_q}\}$ in $K$ called Gauss POINTS or QUADRATURE NODES

  The largest integer $k$ such that $\forall p \in P_k$, $\int_K p(x)\, dx = \sum_{l=1}^{l_q} \omega_l\, p(\xi_l)$ is called the quadrature order and is denoted by $k_q$
- Remark — As regards 1D bounded intervals, the most frequently used quadratures are based on Legendre polynomials which are defined on the interval $(0,1)$ as $\mathcal{E}_k(t) = \frac{1}{k!} \frac{d^k}{dt^k} (t^2 - t)^k$, $k \geq 0$. Note that they are orthogonal on $(0,1)$ with the weight $W = 1$.

## SPECTRAL GALERKIN METHOD — NUMERICAL QUADRATURES (III)

- Theorem — Let $l_q \geq 1$, denote by $\xi_1, \dots, \xi_{l_q}$ the $l_q$ roots of the Legendre polynomial $\mathcal{E}_{l_q}(x)$ and set $\omega_l = \int_0^1 \prod_{\substack{j=1 \\ j \neq l}}^{l_q} \frac{t-\xi_j}{\xi_l-\xi_j}\,dt$. Then $\{\xi_1, \dots, \xi_{l_q}, \omega_1, \dots, \omega_{l_q}\}$ is a quadrature of order $k_q = 2l_q - 1$ on $[0,1]$

  *Proof — Let $\{\mathcal{L}_1, dots, \mathcal{L}_{l_q}\}$ be the set of Lagrange polynomials associated with the Gauß points $\{\xi_1, \dots, \xi_{l_q}\}$. Then $\omega_l = \int_0^1 \mathcal{L}_l(t)\,dt$, $1 \leq l \leq l_q$*

  - *when $p(x)$ is a polynomial of degree less than $l_q$, we integrate both sides of the identity $p(t) = \sum_{l=1}^{l_q} p(\xi_l)\mathcal{L}_l(t)\,dx$, $\forall t \in [0,1]$ and deduce that the quadrature is exact for $p(x)$*
  - *when the polynomial $p(x)$ has degree less than $2l_q$ we write it in the form $p(x) = q(x)\mathcal{E}_{l_q}(x) + r(x)$, where both $q(x)$ and $r(x)$ are polynomials of degree less than $l_q$; owing to orthogonality of the Legendre polynomials, we conclude*

    $$\int_0^1 p(t)\,dt = \int_0^1 r(t)\,dt = \sum_{l=1}^{l_q} \omega_l r(\xi_l) = \sum_{l=1}^{l_q} \omega_l p(\xi_l),$$

    *since the points $\xi_l$ are also roots of $\mathcal{E}_{l_q}$*

## SPECTRAL GALERKIN METHOD — NUMERICAL QUADRATURES (IV)

- PERIODIC GAUSSIAN QUADRATURE — If the interval $[a,b] = [0,2\pi]$ is periodic, the weight $w(x) \equiv 1$ and $P_N(x)$ is the trigonometric polynomial of degree $N$, the Gaussian quadrature is equivalent to the TRAPEZOIDAL RULE (i.e., the quadrature with unit weights and equispaced nodes)
- Evaluation of the spectral coefficients:
  - Assume $\{\phi\}_{k=1}^N$ is a set of basis functions orthogonal under the weight $w$

    $$\hat{u}_k = \int_a^b w(x)u(x)\phi_k(x)\,dx \cong, \sum_{i=0}^{N} w(x_i)u(x_i)\phi_k(x_i), \quad k = 0, \dots, N,$$

    where $x_i$ are chosen so that $\phi_{N+1}(x_i) = 0$, $i = 0, \dots, N$
  - Denoting $\hat{U} = [\hat{u}_0, \dots, \hat{u}_N]^T$ and $U = [u(x_0), \dots, u(x_N)]^T$ we can write the above as

    $$\hat{U} = \mathbb{T}U,$$

    where $\mathbb{T}$ is a TRANSFORMATION MATRIX

## SPECTRAL INTERPOLATION (I)

- INTERPOLATION is a way of determining an expansion of a function $u$ in terms of some ORTHONORMAL BASIS FUNCTIONS alternative to Galerkin spectral projections
- Assuming that $S_N = \text{span}\{e^{i0k}, \dots, e^{iNx}\}$ , we can determine an INTERPOLANT $v \in S_N$ of $u$, such that $v$ coincides with $u$ at $2N+1$ points $\{x_j\}_{|j| \leq N}$ defined by

  $$x_j = jh, \quad |j| \leq N, \quad \text{where} \quad h = \frac{2\pi}{2N+1}$$

- For the interpolant we set

  $$v(x) = \sum_{|k| \leq N} a_k e^{ikx}$$

  where the coefficients $a_k$, $k = 1, \dots, N$ can be determined by solving the algebraic system (cf. page 71)

  $$\sum_{|k| \leq N} e^{ikx_j} a_k = u(x_j), \quad |j| \leq N$$

  with the matrix $\mathbb{A}_{kj} = e^{ikx_j}$, $\quad k, j = 1, \dots, N$

## SPECTRAL INTERPOLATION (II)

- The system can be rewritten as

  $$\sum_{|k| \leq N} W^{jk} a_k = u(x_j), \quad |j| \leq N$$

  where $W = e^{ih} = e^{\frac{2i\pi}{2N+1}}$ is the principal root of order $(2N+1)$ of unity (since $W^{jk} = \left(e^{ih}\right)^{jk}$)
- The matrix $[\mathbb{W}]_{jk} = W^{jk}$ is unitary (i.e. $\mathbb{W}^T\,\overline{\mathbb{W}} = \mathbb{I}(2N+1)$)

  Proof: Examine the expression

  $$\frac{1}{2N+1}\mathbb{W}^T\,\overline{\mathbb{W}} = \mathbb{I} \implies \frac{1}{2N+1}\sum_{|j| \leq N} W^{jk}W^{-jl} = \delta_{kl}$$

  - If $k = l$, then $W^{jk}W^{-jl} = W^{j(k-l)} = W^0 = 1$
  - If $k \neq l$, define $\omega = W^{k-l}$, then

    $$\frac{1}{2N+1}\sum_{|j| \leq N} W^{jk}W^{-jl} = \frac{1}{2N+1}\sum_{|j| \leq N} \omega^j = \frac{1}{M}\sum_{j'=0}^{M-1} \omega^{j'}$$

    where $M = 2N+1$, $j' = j$ if $0 \leq j \leq N$ and $j' = j+M$ if $-N \leq j < 0$, so that $\omega^{j+M} = \omega^j$. Using the expression for the sum of a finite geometric series completes the proof: $(1-\omega)\sum_{j'=0}^{M-1} \omega^{j'} = 1 - \omega^M = 0$

# Spectral Interpolation (III)

- Since the matrix $\mathbb{W}$ is unitary and hence its INVERSE is given by its TRANSPOSE , the Fourier coefficients of the INTERPOLANT of $u$ in $S_N$ can be calculated as follows:

$$a_k = \frac{1}{2N+1}\sum_{|k|\le N} z_j W^{-jk}, \quad \text{where } \; z_j = u(x_j)$$

- The mapping

$$\{z_j\}_{|j|\le N} \longrightarrow \{a_k\}_{|k|\le N}$$

is referred to as Discrete Fourier Transform (DFT)

- Straightforward evaluation of the expressions for $a_k$, $k = 1,\dots,N$ (matrix–vector products) would result in the computational cost $O(N^2)$; clever factorization of this operation, known as the Fast Fourier Transforms (FFT) , reduces this cost down to $O(N\log(N))$

- See `www.fftw.org` for one of the best publicly available implementations of the FFT.

# Spectral Interpolation (IV)

- Let $P_C : C^0_p(I) \to S_N$ be the mapping which associates with $u$ its interpolant $v \in S_N$. Let $(\cdot,\cdot)_N$ be the Gaussian Quadrature approximation of the inner product $(\cdot,\cdot)$

$$(u,v) = \int_{-\pi}^{\pi} u\bar{v}\,dx \cong \frac{1}{2N+1}\sum_{|j|\le N} u(x_j)\overline{v(x_j)} \triangleq (u,v)_N$$

- By construction, the operator $P_C$ satisfies:

$$(P_C u)(x_j) = u(x_j), \quad |j| \le N$$

and therefore also (orthogonality of the defect to $S_N$)

$$(u - P_C u, v_N)_N = 0, \quad \forall v_N \in S_N$$

- By the definition of $P_N$ we have

$$(u - P_N u, v_N) = 0, \quad \forall v_N \in S_N$$

- Thus, $P_C u(x) = \sum_{k=-N}^{N} (u, e^{ikx})_N e^{ikx}$ can be obtained analogously to $P_N u(x) = \sum_{k=-N}^{N} (u, e^{ikx}) e^{ikx}$ by replacing the scalar product $(\cdot,\cdot)$ with the DISCRETE SCALAR PRODUCT $(\cdot,\cdot)_N$

# Spectral Interpolation (V)

- Thus, the INTERPOLATION COEFFICIENTS $a_k$ are equivalent to the Fourier SPECTRAL COEFFICIENTS $\hat{u}_k$ when the latter are evaluated using the Gaussian Quadratures

- The two scalar products coincide on $S_N$, i.e.

$$(u_N, v_N) = (u_N, v_N)_N, \quad \forall u_N, v_N \in S_N,$$

hence for $u \in S_N$, $\hat{u}_k = a_k$, $k = 1,\dots,N$

- Proof — examine the numerical integration formula

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \cong \frac{1}{2N+1}\sum_{|j|\le N} f(x_j);$$

then for every $f = \sum_{k=-N}^{N} \hat{u}_k e^{ikx} \in S_N$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}\,dx = \frac{1}{2N+1}\sum_{|j|\le N} e^{ikx_j} = \frac{1}{2N+1}\sum_{|j|\le N} W^{jk} = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, for the uniform distribution of $x_j$, the Gaussian (trapezoidal) formula is EXACT for $f \in S_N$

# Spectral Interpolation (VI)

- Relation between Fourier coefficients $\hat{u}_k$ of a function $u(x)$ and Fourier coefficients $a_k$ of its interpolant; assume that $u(x) \notin S_N$

$$\hat{u}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\overline{W}_k\,dx, \qquad W_k(x) = e^{ikx}$$

$$a_k = \frac{1}{2N+1}\sum_{|j|\le N} u(x_j)\overline{W_k(x_j)}$$

- *THEOREM* — For $u \in C^0_p(I)$ we have the relation

$$a_k = \sum_{l\in\mathbb{Z}} \hat{u}_{k+lM}, \quad \text{where } M = 2N+1$$

Proof — Consider the set of basis functions (in $L_2(I)$) $U_k = e^{ikx}$. We have:

$$(U_k, U_n)_N = \frac{1}{2N+1}\sum_{|j|\le N} U_k(x_j)\overline{U_n(x_j)} = \frac{1}{2N+1}\sum_{|j|\le N} W^{j(k-n)} = \begin{cases} 1 & k = n \; (mod\ M) \\ 0 & \text{otherwise} \end{cases}$$

Since $P_C u = \sum_{|j|\le N} a_j W_j$, we infer from $(P_C u, W_k)_N = (u, W_k)_N$ that

$$a_k = (P_C u, W_k)_N = (u, W_k)_N = \left(\sum_{n\in\mathbb{Z}} \hat{u}_n W_n, W_k\right)_N = \sum_{n\in\mathbb{Z}} \hat{u}_n (W_n, W_k)_N = \sum_{l\in\mathbb{Z}} \hat{u}_{k+lM}$$

# Spectral Interpolation (VII)

- Thus

$$u(x_j)=v(x_j)=\sum_{k=-\infty}^{\infty}\hat{u}_k e^{ikx_j}=\sum_{|k|\le N}a_k e^{ikx_j}=\sum_{|k|\le N}\left(\hat{u}_k+\sum_{l\in\mathbb{Z}\setminus\{0\}}\hat{u}_{k+lM}\right)e^{ikx_j}$$

- Extremely Important Corollary Concerning Interpolation — two trigonometric polynomials $e^{ik_1x}$ and $e^{ik_2x}$ with different frequencies $k_1$ and $k_2$ are equal at the collocation points $x_j$, $|j|\le N$ when

$$k_2-k_1=l(2N+1),\quad l=0,\pm1,\dots.$$

  Therefore, give a set of values at the collocation points $x_j$, $|j|\le N$, it is impossible to distinguish between $e^{ik_1x}$ and $e^{ik_2x}$. This phenomenon is referred to as aliasing

- Note, however, that the modes appearing in the alias term correspond to frequencies larger than the cut–off frequency $N$.

# Spectral Interpolation (VIII) — Error Estimates in $H_p^s(I)$

- Suppose $s\le r$, $r>\frac{1}{2}$ are given, then there exists a constant $C$ such that if $u\in H_p^r(I)$, we have

$$\|u-P_Cu\|_s\le C(1+N^2)^{\frac{s-r}{2}}\|u\|_r$$

  Outline of the proof:
  Note that $P_C$ leaves $S_N$ invariant, therefore $P_CP_N=P_N$ and we may thus write

$$u-P_Cu=u-P_Nu+P_C(P_N-I)u$$

  Setting $w=(I-P_N)u$ and using the "triangle inequality" we obtain

$$\|u-P_Cu\|_s\le\|u-P_Nu\|_s+\|P_Cw\|_s$$

  - The term $\|u-P_Nu\|_s$ is upper–bounded using theorem from page 91
  - Need to estimate $\|P_Cw\|_s$ — straightforward, but tedious ...

# Spectral Interpolation (IX)

- Until now, we defined the Discrete Fourier Transform for an odd number $(2N+1)$ of grid points
- FFT algorithms generally require an even number of grid points
- We can define the discrete transform for an even number of grid points by constructing the interpolant in the space $\tilde{S}_N$ for which we have $\dim(\tilde{S}_N)=2N$. To do this we choose:

$$\tilde{x}_j=j\tilde{h},\qquad -N+1\le j\le N$$
$$\tilde{h}=\frac{\pi}{N}$$

- All results presented before can be established in the case with $2N$ grid points with only minor modifications
- However, now the $N$-th Fourier mode $\hat{u}_N$ does not have its complex conjugate! This coefficient is usually set to zero ($\hat{u}_N=0$) to avoid an uncompensated imaginary contribution resulting from differentiation
- odd or even collocation depending on whether $M=2N+1$ or $M=2N$

# Spectral Interpolation (X)

- Before we focused on representing the Interpolant as a Fourier series $v(x_j)=\sum_{k=-N}^{N}a_ke^{ikx_j}$
- Alternatively, we can represent the Interpolant using the nodal values as (assuming, for the moment, infinite domain $x\in\mathbb{R}$)

$$v(x)=\sum_{j=-\infty}^{\infty}u(x_j)C_j(x),$$

  where $C_j(x)$ is a cardinal function with the property that $C_j(x_i)=\delta_{ij}$ (i.e., generalization of the Lagrange Polynomial for infinite domain)

- In an infinite domain we have the Whittaker Cardinal or sinc function

$$C_k(x)=\frac{\sin[\pi(x-kh)/h]}{\pi(x-kh)/h}=\mathrm{sinc}[(x-kh)/h],$$

  where $\mathrm{sinc}(x)=\frac{\sin(\pi x)}{\pi x}$

- Proof — the Fourier transform of $\delta_{j0}$ is $\hat{\delta}(k)=h$ for all $k\in[-\pi/h,\pi/h]$; hence, the interpolant of $\delta_{j0}$ is $v(x)=\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h}e^{ikh}\,dk=\frac{\sin(\pi x/h)}{\pi x/h}$

# Spectral Interpolation (XI)

- Thus, the spectral interpolant of a function in an INFINITE domain is a linear combination of WHITTAKER CARDINAL functions
- In a PERIODIC DOMAIN we still have the representation

  $$v(x) = \sum_{j=0}^{N-1} u(x_j) S_j(x),$$

  but now the CARDINAL FUNCTIONS have the form

  $$S_j(x) = \frac{1}{N} \sin\left[\frac{N(x-x_j)}{2}\right] \cot\left[\frac{(x-x_j)}{2}\right]$$

- Proof — similar to the previous (unbounded) case, except that now the interpolant in given by a DISCRETE Fourier Transform
- The relationship between the Cardinal Functions corresponding to the PERIODIC and UNBOUNDED domains

  $$S_0(x) = \frac{1}{2N} \sin(Nx) \cot(x/2) = \sum_{m=-\infty}^{\infty} \operatorname{sinc}\left(\frac{x-2\pi m}{h}\right)$$

# Spectral Differentiation (I)

- Two ways to calculate the derivative $w(x_j) = u'(x_j)$ based on the values $u(x_j)$, where $0 \le j \le 2N+1$; denote $U = [u_0, \dots, u_{2N+1}]^T$ and $U' = [u'_0, \dots, u'_{2N+1}]^T$
- METHOD ONE — approach based on differentiation in Fourier space:
  - calculate the vector of Fourier coefficients $\hat{U} = \mathbb{T} U$
  - apply the diagonal differentiation matrix $\hat{U}' = \hat{\mathbb{D}} \hat{U}$ (cf. page 94)
  - return to real space via inverse Fourier transform $U = \mathbb{T}^T \hat{U}$
- REMARK — formally we can write

  $$U' = \mathbb{T}^T \hat{\mathbb{D}} \mathbb{T} U,$$

  however in practice matrix operations are replaced by FFTs

# Spectral Differentiation (II)

- METHOD TWO — approach based on differentiation (in real space) of the interpolant $u'(x_j) = v'(x_j) = \sum_{j=0}^{N-1} u(x_j) S'_j(x)$, where the cardinal function has the following derivatives

  $$S'(x_j) = \begin{cases} 0, & j = 0 \ (mod\ N) \\ \frac{1}{2}(-1)^j \cot(jh/2), & j \neq 0 \ (mod\ N) \end{cases}$$

- Thus, since the interpolant is a linear combination of shited Cardinam Functions, the differentiation matrix has the form of a TOEPLITZ CIRCULANT matrix

  $$\mathbb{D} = \begin{bmatrix} 0 & & & -\frac{1}{2}\cot[(1h)/2] \\ -\frac{1}{2}\cot[(1h)/2] & \ddots & \ddots & \frac{1}{2}\cot[(2h)/2] \\ \frac{1}{2}\cot[(2h)/2] & \ddots & & -\frac{1}{2}\cot[(3h)/2] \\ -\frac{1}{2}\cot[(3h)/2] & & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{2}\cot[(1h)/2] \\ \frac{1}{2}\cot[(1h)/2] & & & 0 \end{bmatrix}$$

- Higher–order derivatives obtained calculating $S^{(p)}(x_j)$