Solution of a Model Elliptic Problem (I)

- We are interested in a Partial Differential Equation (a boundary value problem) of the general form $\mathcal{L}u = f$
- We will look for solutions in the form:

$$u_N(x) = \sum_{|k|\le N} \hat{u}_k e^{ikx}, \qquad = \sum_{j=1}^{2N+1} u(x_j) S_j(x),$$

  where $S_j(x)$ is the periodic cardinal function centered at x_j
- For the above model problem we will analyze:
  - spectral Galerkin method
  - spectral Collocation method
    - variant with the Fourier coefficients $\hat{u}_k$ as the unknowns
    - variant with the nodal values $u(x_j)$ as the unknowns

Solution of a Model Elliptic Problem (II)

- Consider the following 1D second–order elliptic problem in a periodic domain $\Omega = [0, 2\pi]$

$$\mathcal{L}u \triangleq \nu u'' - au' + bu = f,$$

  where ν, a and b are constant and $f = f(x)$ is a smooth 2π–periodic function.
- For $\nu = 10$, $a = 1$, $b = 5$ and the RHS function

$$f(x) = e^{\sin(x)} \left[\nu(\cos^2(x) - \sin(x)) - a\cos(x) + b \right]$$

  the solution is

$$u(x) = e^{\sin(x)}$$

- For the Galerkin approach we are interested in 2π–periodic solutions in the form

$$u_N(x) = \sum_{|k|\le N} \hat{u}_k e^{ikx}$$

Solution of an Elliptic Problem — Galerkin Approach (I)

- Residual

$$R_N(x) = \mathcal{L}u_N - f = \sum_{|k|\le N} \hat{u}_k \mathcal{L}e^{ikx} - f$$

- Cancellation of the residual in the mean (setting the projections on the basis functions $W_n(x) = e^{inx}$ equal to zero)

$$(R_N, W_n) = \sum_{k=-N}^{N} \hat{u}_k (\mathcal{L}e^{ikx}, e^{inx}) - (f, e^{inx}) = 0, \quad n = -N, \dots, N$$

- Noting that $\mathcal{L}e^{ikx} = (-\nu k^2 - iak + b)e^{ikx} \triangleq \mathcal{G}_k e^{ikx}$ we obtain

$$\sum_{k=-N}^{N} \mathcal{G}_k \hat{u}_k \int_0^{2\pi} e^{i(k-n)}\, dx = \hat{f}_n, \quad n = -N, \dots, N$$

- Assuming $\mathcal{G}_k \neq 0$, we obtain the Galerkin equations for $\hat{u}_k$

$$\mathcal{G}_k \hat{u}_k = \hat{f}_k, \qquad k = -N, \dots, N$$

  - The Galerkin equations are decoupled
  - Since u is real, it is necessary to calculate $\hat{u}_k$ for $k \ge 0$ only

Solution of an Elliptic Problem — Collocation Approach (I)

- Residual (with the expansion coefficients $\hat{u}_k$ as unknowns)

$$R_N(x) = \mathcal{L}u_N - f = \sum_{|k|\le N} \hat{u}_k \mathcal{L}e^{ikx} - f$$

- Canceling the residual pointwise at the collocation points x_j, $j = 1, \dots, M$

$$\sum_{k=-N}^{N} (\mathcal{G}_k \hat{u}_k - \tilde{f}_k) e^{ikx_j} = 0, \quad j = 1, \dots, M$$

  where (note the aliasing error) $\tilde{f}_k = \hat{f}_k + \sum_{l\in\mathbb{Z}\setminus\{0\}} \hat{f}_{k+lM}$
- Thus, the collocation equations for the Fourier coefficients

$$\mathcal{G}_k \hat{u}_k = \tilde{f}_k = \hat{f}_k + \sum_{l\in\mathbb{Z}\setminus\{0\}} \hat{f}_{k+lM}, \quad k = -N, \dots, N$$

  - Formally, the Galerkin and collocation methods are distinct
  - In practice, the projection (f, e^{ikx}) is evaluated using FFT and therefore also involves aliasing errors. Therefore, for the present problem, the two approaches are numerically equivalent .

SOLUTION OF AN ELLIPTIC PROBLEM — COLLOCATION APPROACH (II)

- RESIDUAL (with the nodal values $u_N(x_j)$, $j = 1,\dots,M$, as unknowns)

$$R_N(x) = \mathcal{L}u_N - f$$

- Canceling the residual pointwise at the collocation points x_j, $j = 1,\dots,M$

$$[R_N(x_1),\dots,R_N(x_M)]^T = \mathbb{L}U_N - F = (\nu\mathbb{D}_2 - a\mathbb{D}_1 + b\mathbb{I})U_N - F = 0,$$

  where $U_N = [u_N(x_1),\dots,u_N(x_M)]^T$ and $\mathbb{D}_1$ and $\mathbb{D}_2$ are the differentiation matrices.

- Derivation of the DIFFERENTIATION MATRICES

$$\left.\begin{aligned} u_N^{(p)}(x_j) &= \sum_k (ik)^p \hat{u}_k e^{ikx_j} \\ \hat{u}_k &= \frac{1}{M}\sum_{j=1}^{M} u_N(x_j)e^{-ikx_j} \end{aligned}\right\} \Longrightarrow u_N^{(p)}(x_i) = \sum_{j=1}^{M} d_{ij}^{(p)} u_N(x_j)$$

SOLUTION OF AN ELLIPTIC PROBLEM — COLLOCATION APPROACH (III)

- Differentiation Matrices (for even collocation, i.e., $I_N = -N+1,\dots,N$ and $M = 2N$)

$$d_{ij}^{(1)} = \begin{cases} \frac{1}{2}(-1)^{i+j}\cot(h_{ij}) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \qquad d_{ij}^{(2)} = \begin{cases} \frac{1}{4}(-1)^{i+j}N + \frac{(-1)^{i+j+1}}{2\sin^2(h_{ij})} & \text{if } i \neq j \\ -\frac{(N-1)(N-2)}{12} & \text{if } i = j \end{cases}$$

- Remarks:
  - The differentiation matrices are full (and not so well–conditioned ...), so the system of equations for $u_N(x_j)$ is now COUPLED
  - For constant coefficient PDEs the present approach is therefore inferior to the first collocation approach with the Fourier coefficients used as unknowns
  - Note the relationship to the banded matrices obtained when approximating differential operators using finite differences
- QUESTION — Derive the above differentiation matrices, also for the case of odd collocation

NYQUIST–SHANNON SAMPLING THEOREM

- If a periodic function $f(x)$ has a Fourier transform $\hat{f}_k = 0$ for $|k| > M$, then it is completely determined by providing the function values at a series of points spaced $\Delta x = \frac{1}{2M}$ apart. The values $f_n = f(\frac{n}{2M})$ are called the SAMPLES OF $f(x)$.
- The minimum sampling frequency that allows for reconstruction of the original signal, that is $2M$ samples per unit distance, is known as the NYQUIST FREQUENCY . The time in between samples is called the NYQUIST INTERVAL .
- The NYQUIST–SHANNON SAMPLING THEOREM is a fundamental tenet in the field of INFORMATION THEORY (originally formulated by Nyquist in 1928, but formally proved by Shannon only in 1949)

PDES WITH VARIABLE COEFFICIENTS — GALERKIN APPROACH (I)

- Consider again the problem $\mathcal{L}u = \nu u'' - au' + bu = f$, but assume now that the coefficient a is a function OF SPACE $a = a(x)$
- The following Galerkin equations are obtained for $\hat{u}_k$

$$-\nu k^2 \hat{u}_k - i\sum_{p=-N}^{N} p\hat{a}_{k-p}\hat{u}_p + b\hat{u}_k = \hat{f}_k, \quad k = -N,\dots,N,$$

  where $a(x) \cong a_N(x) = \sum_{k=-N}^{N}\hat{a}_k e^{ikx}$ and $f(x) \cong f_N(x) = \sum_{k=-N}^{N}\hat{f}_k e^{ikx}$; Note that

$$\sum_{q=-N}^{N}\hat{a}_q e^{iqx}\sum_{p=-N}^{N}\hat{u}_p e^{ipx} = \sum_{q,p=-N}^{N}\hat{a}_q\hat{u}_p e^{i(q+p)x} = \sum_{k=-2N}^{2N}\sum_{\substack{q,p=-N \\ q+p=k}}^{N}\hat{a}_q\hat{u}_p e^{ikx}$$

$$= \sum_{k=-2N}^{2N}\sum_{p=-N}^{N}\hat{a}_{k-p}\hat{u}_p e^{ikx}, \text{ where } \hat{a}_q, \hat{u}_q \equiv 0, \text{ for } |q| > N$$

- Now the Galerkin equations are COUPLED (a system of equations has to be solved)

PDEs with variable coefficients — Collocation Approach (I)

- With Fourier coefficients $\hat{u}_k$ as unknowns, the collocation equations are

$$-\sum_{k=-N}^{N}(\nu k^2+b)\hat{u}_k e^{ikx_j} - a(x_j)\sum_{k=-N}^{N} ik\hat{u}_k e^{ikx_j} = f(x_j),\quad j=1,\dots,M$$

- Approximations of the Fourier coefficients of $a(x)$ and $f(x)$, $\hat{a}_k^c$ and $\hat{f}_k^c$, respectively, are calculated using the Discrete Fourier Transform;

$$a(x_j)\sum_{k=-N}^{N} ik\hat{u}_k e^{ikx_j} = \sum_{p=-N}^{N}\hat{a}_p^c e^{ipx_j}\sum_{q=-N}^{N} iq\hat{u}_q e^{iqx_j} =$$

$$i\sum_{k=-N}^{N}\left(\sum_{\substack{q,p=-N\\ q+p=k}}^{N} q\hat{a}_p^c\hat{u}_q + \sum_{\substack{q,p=-N\\ q+p=k+N}}^{N} q\hat{a}_p^c\hat{u}_q + \sum_{\substack{q,p=-N\\ q+p=k-N}}^{N} q\hat{a}_p^c\hat{u}_q\right)e^{ikx_j}$$

$$\triangleq i\sum_{k=-N}^{N}\hat{S}_k e^{ikx_j}$$

- The resulting algebraic system is

$$-\nu k^2\hat{u}_k - i\hat{S}_k + b\hat{u}_k = \hat{f}_k,\quad k=-N,\dots,N,$$

PDEs with variable coefficients — Collocation Approach (II)

- Expressing (hypothetically) $a(x)$ and $f(x)$ with infinite Fourier series we obtain

$$au'\Big|_{x=x_j} = i\sum_{k=-N}^{N}(\hat{S}_k^{(0)}+\hat{S}_k^{(1)}+\hat{S}_k^{(2)}+\hat{S}_k^{(3)})e^{ikx_j}$$

$$= i\sum_{k=-N}^{N}\left(\sum_{\substack{q,p=-N\\ q+p=k}}^{N} q\hat{a}_p^c\hat{u}_q + \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{\substack{q,p=-N\\ q+p=k}}^{N} q\hat{a}_{p+mM}^c\hat{u}_q\right.$$

$$\left.+\sum_{m=-\infty}^{\infty}\sum_{\substack{q,p=-N\\ q+p=k+N}}^{N} q\hat{a}_{p+mM}^c\hat{u}_q + \sum_{m=-\infty}^{\infty}\sum_{\substack{q,p=-N\\ q+p=k-N}}^{N} q\hat{a}_{p+mM}^c\hat{u}_q\right)$$

- The collocation equation become

$$-\nu k^2\hat{u}_k - i\hat{S}_k^{(0)} + i\left(\hat{S}_k^{(1)}+\hat{S}_k^{(2)}+\hat{S}_k^{(3)}\right) + b\hat{u}_k = \hat{f}_k^e + \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\hat{f}_{k+mM}^e,\quad k=-N,\dots,N,$$

- Note that the terms in red are absent in the corresponding Galerkin formulation ; hence the two approaches are not numerically equivalent anymore.

PDEs with variable coefficients — Collocation Approach (III)

- With the nodal values $u(x_j)$, $j=1,\dots,M$ as unknowns, the collocation equations are (cf. 117)

$$(\nu\mathbb{D}_2 - \mathbb{D}' + b\mathbb{I})U_N = F,$$

  where the matrix $\mathbb{D}' = \left[a(x_j)d_{jk}^{(1)}\right]$, $j,k=1,\dots,M$

- Again, solution of an algebraic system is required

Fourier transforms in higher dimensions

- Consider a function $u=u(x,y)$ 2π–periodic in both x and y;
  Direct Discrete Fourier Transform

$$\hat{u}_{k_x,k_y} = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi}u(x,y)e^{-ik_x x}dx\right]e^{-ik_y y}dy = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}u(x,y)e^{-i\mathbf{k}\cdot\mathbf{r}}dxdy,$$

  where $\mathbf{k}=[k_x,k_y]$ is the wavevector and $\mathbf{r}=[x,y]$ is the position vector.

- Representation of a function $u=u(x,y)$ as a double Fourier series

$$u(x,y) = \sum_{k_x,k_y=-N}^{N}\hat{u}_{k_x,k_y}e^{i(k_x x+k_y y)} = \sum_{k_x,k_y=-N}^{N}\hat{u}_{k_x,k_y}e^{i\mathbf{k}\cdot\mathbf{r}}$$

- Fourier transforms in two (and more) dimensions can be efficiently performed using most standard FFT packages.

NONLINEAR EVOLUTION PDES

- Replacing the term au' with the NONLINEAR the term uu' and applying Galerkin or collocation method leads to a SYSTEM OF NONLINEAR EQUATIONS that need to be solved using iterative techniques
- From now on we will focus on TIME–DEPENDENT (evolution) PDEs and as a model problem will consider the BURGERS EQUATION

$$\begin{cases} \partial_t u + u\partial_x u - \nu\partial_{xx} u = 0 & \text{in } [0,2\pi]\times[0,T] \\ u(x) = u_0(x) & \text{at } t = 0 \end{cases}$$

  Note that steady problems can sometimes be solved as a steady limit of certain time–dependent problems.
- Looking for solution in the form

$$u_N(x,t) = \sum_{k=-N}^{N} \hat{u}_k(t) e^{ikx}$$

  Note that the expansion coefficients $\hat{u}_k(t)$ are now FUNCTIONS OF TIME
- Denote by u_N^n the approximation of u_N at time $t_n = n\Delta t$, $n = 0,1,\dots$

NONLINEAR EVOLUTION PDES — GALERKIN APPROACH (I)

- Time–discretization of the residual $R_N(x,t)$

$$R_N^n = \frac{u_N^{n+1} - u_N^n}{\Delta t} + u_N^n \partial_x u_N^n - \nu \partial_{xx} u_N^{n+1}$$

  Points to note:
  - EXPLICIT treatment of the nonlinear term avoids costly iterations
  - IMPLICIT treatment of the linear viscous term allows one to mitigate the stability restrictions on the time step Δt
  - here using for simplicity first–order accurate explicit/implicit Euler — can do much better than that
- system of equations obtained by applying the GALERKIN FORMALISM

$$\left(\frac{1}{\Delta t} + \nu k^2\right)\hat{u}_k^{n+1} = \frac{1}{\Delta t}\hat{u}_k^n - i \sum_{\substack{p,q=-N \\ p+q=k}}^{N} q\hat{u}_p^n \hat{u}_q^n, \quad k = -N,\dots,N$$

  Note truncation of higher modes in the nonlinear term.

NONLINEAR EVOLUTION PDES — GALERKIN APPROACH (II)

- Evaluation of the nonlinear $i\sum_{\substack{p,q=-N \\ p+q+k}}^{N} q\hat{u}_p^n \hat{u}_q^n$ term in Fourier space results in a CONVOLUTION SUM which requires $O(N^2)$ operations – can we do better that that?
- PSEUDOSPECTRAL APPROACH — perform differentiation in FOURIER SPACE and evaluate products in REAL SPACE ; transition between the two representations is made using FFTs which cost "only" $O(N\log(N))$
  Outline of the algorithm:
  1. calculate (using inverse FFT) $u_N^n(x_j)$, $j = 1,\dots,M$ from $\hat{u}_k^n$, $k = -N\dots,N$,
  2. calculate (using inverse FFT) $\partial_x u_N^n(x_j)$, $j = 1,\dots,M$ from $ik\hat{u}_k^n$, $k = -N\dots,N$,
  3. calculate the product $w_N^n(x_j) = u_N^n(x_j)\partial_x u_N^n(x_j)$, $j = 1,\dots,M$
  4. Calculate (using FFT) $\tilde{w}_k^n$, $k = -N\dots,N$ from $w_N^n(x_j)$, $j = 1,\dots,M$
- Note that, because of the ALIASING PHENOMENON , the quantity $\tilde{w}_k^n$ is different from $\hat{w}_k^n = i\sum_{\substack{p,q=-N \\ p+q=k}}^{N} q\hat{u}_p^n \hat{u}_q^n$

NONLINEAR EVOLUTION PDES — GALERKIN APPROACH (III)

- Analysis of aliasing in the PSEUDOSPECTRAL calculation of the nonlinear term

$$w_N^n(x_j) = \sum_{k=-N}^{N} \tilde{w}_k^n e^{ikx_j}, \quad \text{where} \quad w_N^n(x_j) = u_N^n(x_j)\partial_x u_N^n(x_j)$$

  The Discrete Fourier Transform

$$\tilde{w}_k^n = \frac{1}{M}\sum_{j=1}^{M} w_N^n(x_j) e^{-ikx_j} = \frac{1}{M}\sum_{j=1}^{M}\left(\sum_{p=-N}^{N}\hat{u}_p^n e^{ipx_j}\right)\left(\sum_{q=-N}^{N} iq\hat{u}_q^n e^{iqx_j}\right) e^{-ikx_j}$$

$$= \frac{1}{M}\sum_{j=1}^{M}\sum_{p,q=-N}^{N} iq\,\hat{u}_p^n\,\hat{u}_q^n\, e^{i(p+q-k)x_j} = \frac{1}{M}\sum_{p,q=-N}^{N} iq\,\hat{u}_p^n\,\hat{u}_q^n \sum_{j=1}^{M} e^{i(p+q-k)x_j}$$

$$= \hat{w}_k^n + i\sum_{\substack{p,q=-N \\ p+q=k+M}}^{N} q\hat{u}_p^n\hat{u}_q^n + i\sum_{\substack{p,q=-N \\ p+q=k-M}}^{N} q\hat{u}_p^n\hat{u}_q^n \quad k = -N\dots,N$$

  The term $\hat{w}_k^n$ is the convolution sum obtained by TRUNCATING the fully spectral Galerkin approach. The terms IN RED are the ALIASING ERRORS .
- Thus, the PSEUDOSPECTRAL GALERKIN equations are

$$\left(\frac{1}{\Delta t} + \nu k^2\right)\hat{u}_k^{n+1} = \frac{1}{\Delta t}\hat{u}_k^n - \tilde{w}_k^n, \quad k = -N,\dots,N$$

NONLINEAR EVOLUTION PDES — COLLOCATION APPROACH (I)

- Looking for the solution in the form

$$u_N(x,t) = \sum_{k=-N}^{N} \hat{u}_k(t) e^{ikx},$$

  i.e., with the Fourier coefficients $\hat{u}_k$ as unknowns

- Time–discretization of the residual $R_N(x,t)$

$$R_N^n = \frac{u_N^{n+1} - u_N^n}{\Delta t} + u_N^n \partial_x u_N^n - \nu \partial_{xx} u_N^{n+1}$$

- Canceling the residual at the collocation points x_j

$$\frac{1}{\Delta t}\left[u_N^{n+1}(x_j) - u_N^n(x_j)\right] + u_N^n(x_j)\partial_X u_N^n(x_j) - \nu\partial_{xx} u_N^{n+1}(x_j) = 0 \quad j = 1,\dots,M$$

- Straightforward calculation shows that the equation for the Fourier coefficients $\hat{u}_k$ is the same as in the PSEUDOSPECTRAL GALERKIN APPROACH . Thus the two methods are numerically equivalent.

- QUESTION — Show equivalence of pseudospectral Galerkin and collocation approaches to a nonlinear PDE

NONLINEAR EVOLUTION PDES — ALIASING REMOVAL (I)

- "3/2 RULE" — extend the wavenumber range (the "spectrum"), and therefore also the number of collocation points, of the quantities involved in the products, so that the aliasing errors arising in pseudospectral calculations are not present.

- ALGORITHM — consider two 2π–periodic functions

$$a_N(x) = \sum_{k=-N}^{N} \hat{a}_k e^{ikx}, \qquad b_N(x) = \sum_{k=-N}^{N} \hat{b}_k e^{ikx}$$

  Calculated in a naive way, the Fourier coefficients of the product $w(x) = a(x)b(x)$ are

$$\tilde{w}_k = \hat{w}_k + \sum_{\substack{p,q=-N \\ p+q=k+M}}^{N} \hat{a}_p \hat{b}_q + \sum_{\substack{p,q=-N \\ p+q=k-M}}^{N} \hat{a}_p \hat{b}_q, \quad k = -N,\dots,N$$

  where $\hat{w}_k$ are the coefficients of the truncated convolution sum that we want to keep (only)

NONLINEAR EVOLUTION PDES — ALIASING REMOVAL (II)

1. Extend the spectra $\hat{a}_k$ and $\hat{b}_k$ to $\hat{a}'_k$ and $\hat{b}'_k$ according to

$$\hat{a}'_k = \begin{cases} \hat{a}_k & \text{if } |k| \le N \\ 0 & \text{if } N < |k| \le N' \end{cases}, \qquad \hat{b}'_k = \begin{cases} \hat{b}_k & \text{if } |k| \le N \\ 0 & \text{if } N < |k| \le N' \end{cases}$$

   The number N' will be determined later.

2. Calculate (via FFT) $a_{N'}$ and $b_{N'}$ in real space on the extended grid $x'_j = \frac{2\pi j}{M'}$, $j = 1,\dots,M'$, where $M' = 2N'+1$

$$a_{N'}(x'_j) = \sum_{k=-N'}^{N'} \hat{a}'_k e^{ikx'_j}, \qquad b_{N'}(x'_j) = \sum_{k=-N'}^{N'} \hat{b}'_k e^{ikx'_j}$$

3. Multiply $a_{N'}(x'_j)$ and $b_{N'}(x'_j)$: $w'(x'_j) = a_{N'}(x'_j)\, b_{N'}(x'_j)$, $j = 1,\dots,M'$

4. Calculate (via FFT) the Fourier coefficients of $w'(x'_j)$

$$\tilde{w}'_k = \frac{1}{M'} \sum_{j=1}^{M'} w(x'_j) e^{-ikx'_j}, \quad k = -N',\dots,N', \quad M' = 2N'+1$$

   Taking the latter quantity for $k = -N,\dots,N$ gives an expression for the convolution sum FREE OF ALIASING ERRORS

NONLINEAR EVOLUTION PDES — ALIASING REMOVAL (III)

- Making a suitable choice for N'

$$\tilde{w}'_k = \hat{w}_k + \sum_{\substack{p,q=-N' \\ p+q=k+M'}}^{N'} \hat{a}'_p \hat{b}'_q + \sum_{\substack{p,q=-N' \\ p+q=k-M'}}^{N'} \hat{a}'_p \hat{b}'_q$$

$$= \hat{w}_k + \sum_{\substack{p,q=-N \\ p+q=k+M'}}^{N} \hat{a}_p \hat{b}_q + \sum_{\substack{p,q=-N \\ p+q=k-M'}}^{N} \hat{a}_p \hat{b}_q$$

  because $\hat{a}'_p, \hat{b}'_q = 0$ for $|p|,|q| > N$

- The alias terms will vanish, when one of the frequencies p or q appearing in each term of the sum is larger than N. Observe that in the first alias term $q = M' + k - p = 2N' + 1 + k - p$, therefore

$$\min_{|k|,|p| \le N}(q) = \min_{|k|,|p| \le N}(2N' + 1 + k - p) = 2N' + 1 - 2N > N$$

  Hence $2N' > 3N - 1$. One may take $N' \ge 3N/2$ (THE "3/2 RULE")

- Analogous argument for the second aliasing error sum.

Hybrid Integration Schemes for ODEs with both linear and nonlinear terms)

- Consider a model ODE problem

$$\mathbf{y}' = \mathbf{r}(\mathbf{y}) + A\mathbf{y}$$

- One would like to use a higher–order ODE integrator with
  - EXPLICIT treatment of nonlinear terms
  - IMPLICIT treatment of linear terms (with high–order derivatives)
- Combining a three–step Runge–Kutta method with the CRANK–NICHOLSON METHOD results in the following approach:

$$\left(I - \frac{h_{rk}}{2}A\right)\mathbf{y}^{rk+1} = \mathbf{y}^{rk} + \frac{h_{rk}}{2}A\mathbf{y}^{rk} + h_{rk}\beta_{rk}\mathbf{r}(\mathbf{y}^{rk}) + h_{rk}\zeta_{rk}\mathbf{r}(\mathbf{y}^{rk-1}), \quad rk = 1,2,3$$

where

$$h_1 = \frac{8}{15}\Delta t \qquad h_2 = \frac{2}{15}\Delta t \qquad h_3 = \frac{1}{3}\Delta t$$

$$\beta_1 = 1 \qquad \beta_2 = \frac{25}{8} \qquad \beta_3 = \frac{9}{4}$$

$$\zeta_1 = 0 \qquad \zeta_2 = -\frac{17}{8} \qquad \zeta_3 = -\frac{5}{4}$$