CHEBYSHEV POLYNOMIALS — REVIEW (I)

- General properties of ORTHOGONAL POLYNOMIALS
  - Suppose $I = [a,b]$ is a given interval. Let $\omega : I \to \mathbb{R}^+$ be a weight function which is positive and continuous on I
  - Let $L^2_\omega(I)$ denote the space of measurable functions v such that

$$\|v\|_\omega = \left(\int_I |v(x)|^2 \omega(x)\, dx\right)^{\frac{1}{2}} < \infty$$

  - $L^2_\omega(I)$ is a Hilbert space with the scalar products

$$(u,v)_\omega = \int_I u(x)\overline{v(x)}\omega(x)\, dx$$

- CHEBYSHEV POLYNOMIALS are obtained by setting:
  - the weight: $\omega(x) = (1-x^2)^{-\frac{1}{2}}$
  - the interval: $I = [-1,1]$
  - Chebyshev polynomials of degree k are expressed as

$$T_k(x) = \cos(k \cos^{-1} x), \quad k = 0,1,2,\ldots$$

CHEBYSHEV POLYNOMIALS — REVIEW (II)

- By setting $x = \cos(z)$ we obtain $T_k = \cos(kz)$, therefore we can derive expressions for the first Chebyshev polynomials

$$T_0 = 1, \;\; T_1 = \cos(z) = x, \;\; T_2 = \cos(2z) = 2\cos^2(z) - 1 = 2x^2 - 1, \;\; \ldots$$

- More generally, using the de Moivre formula, we obtain

$$\cos(kz) = \Re\left[(\cos(z) + i\sin(z))^k\right],$$

  from which, invoking the binomial formula, we get

$$T_k(x) = \frac{k}{2}\sum_{m=0}^{[k/2]} (-1)^m \frac{(k-m-1)!}{m!(k-2m)!}(2x)^{k-2m},$$

  where $[\alpha]$ represents the integer part of α

- Note that the above expression is COMPUTATIONALLY USELESS — one should use the formula $T_k(x) = \cos(k\cos^{-1} x)$ instead!

CHEBYSHEV POLYNOMIALS — REVIEW (III)

- The trigonometric identity $\cos(k+1)z + \cos(k-1)z = 2\cos(z)\cos(kz)$ results in the following RECURRENCE RELATION

$$2xT_k = T_{k+1} + T_{k-1}, \quad k \ge 1,$$

  which can be used to deduce T_k, $k \ge 2$ based on T_0 and T_1 only

- Similarly, for the derivatives we get

$$T'_k = \frac{d}{dz}(\cos(kz))\frac{dz}{dx} = \frac{d}{dz}(\cos(kz))\left(\frac{dx}{dz}\right)^{-1} = k\frac{\sin(kz)}{\sin(z)},$$

  which, upon using trigonometric identities, yields a RECURRENCE RELATION for derivatives

$$2T_k = \frac{T'_{k+1}}{k+1} - \frac{T'_{k-1}}{k-1}, \quad k > 1,$$

CHEBYSHEV POLYNOMIALS — REVIEW (IV)

- Note that simply changing the integration variable we obtain

$$\int_{-1}^{1} f(x)\omega(x)\, dx = \int_0^{\pi} f(\cos\theta)\, d\theta$$

  This also provides an isometric (i.e., norm–preserving) transformation $u \in L^2_\omega(I) \longrightarrow \tilde{u} \in L^2(0,\pi)$, where $\tilde{u}(\theta) = u(\cos\theta)$

- Consequently, we obtain

$$(T_k, T_l)_\omega = \int_{-1}^{1} T_k T_l \omega\, dx = \int_0^{\pi} \cos(k\theta)\cos(l\theta)\, d\theta = \frac{\pi}{2} c_k \delta_{kl},$$

  where

$$c_k = \begin{cases} 2 & \text{if } k = 0, \\ 1 & \text{if } k \ge 1 \end{cases}$$

- Note that Chebyshev polynomials are ORTHOGONAL , but not ORTHONORMAL

CHEBYSHEV POLYNOMIALS — REVIEW (V)

- The Chebyshev polynomials $T_k(x)$ vanish at the GAUSS POINTS x_j defined as

$$x_j = \cos\left(\frac{(2j+1)\pi}{2k}\right), \quad j = 0,\dots,k-1$$

There are exactly k distinct zeros in the interval $[-1,1]$

- Note that $-1 \le T_k \le 1$; furthermore the Chebyshev polynomials $T_k(x)$ attain their extremal values at the the GAUSS–LOBATTO POINTS x_j defined as

$$x_j = \cos\left(\frac{j\pi}{k}\right), \quad j = 0,\dots,k$$

There are exactly $k+1$ real extrema in the interval $[-1,1]$.

CHEBYSHEV POLYNOMIALS — CLUSTERED GRIDS (I)

- Interpolation on CLUSTERED GRIDS has very special properties — CHEBYSHEV MINIMAL AMPLITUDE THEOREM : Of all polynomials of degree N with the leading coefficient (i.e., the coefficient of x^N) equal to 1, the unique polynomial which has the smallest maximum on $[-1,1]$ is $T_N(x)/2^{N-1}$, the N–th Chebyshev polynomials divided by 2^{N-1}. In other words, all polynomials of the same degree and leading coefficient satisfy the inequality

$$\max_{x\in[-1,1]} |P_N(x)| \ge \max_{x\in[-1,1]} \left|\frac{T_N(x)}{2^{N-1}}\right| = \frac{1}{2^{N-1}}$$

- Hence, the TRUNCATION ERROR when given in terms of $\frac{1}{2^N} T_{N+1}(x)$ will be best behaved
- Thus, in contrast to interpolation on UNIFORM grids, interpolation on CLUSTERED grid is less likely to exhibit the RUNGE PHENOMENON ; this concerns clustered grids with asymptotic density of points proportional to $\frac{N}{\pi\sqrt{1-x^2}}$ (e.g., various Chebyshev grids)

CHEBYSHEV POLYNOMIALS — NUMERICAL INTEGRATION FORMULAE (I)

- FUNDAMENTAL THEOREM OF GAUSSIAN QUADRATURE — The abscissas of the N–point Gaussian quadrature formula are precisely the roots of the orthogonal polynomial of order N for the same interval and weighting function.
- THE GAUSS–CHEBYSHEV FORMULA (exact for $u \in \mathbb{P}_{2N-1}$)

$$\int_{-1}^{1} u(x)\omega(x)\,dx = \frac{\pi}{N}\sum_{j=1}^{N} u(x_j),$$

with $x_j = \cos\left(\frac{(2j-1)\pi}{2N}\right)$ (the Gauss points located in the interior of the domain only)
Proof via straightforward application of the theorem quoted above.

CHEBYSHEV POLYNOMIALS — NUMERICAL INTEGRATION FORMULAE (II)

- THE GAUSS–RADAU–CHEBYSHEV FORMULA (exact for $u \in \mathbb{P}_{2N}$)

$$\int_{-1}^{1} u(x)\omega(x)\,dx = \frac{\pi}{2N+1}\left[u(\xi_0) + 2\sum_{j=1}^{N} u(\xi_j)\right],$$

with $\xi_j = \cos\left(\frac{2j\pi}{2N+1}\right)$ (the Gauss–Radau points located in the interior of the domain and on one boundary, useful e.g., in annular geometry)
Proof via application of the above theorem and using the roots of the polynomial $Q_{N+1}(x) = T_N(a)T_{N+1}(x) - T_{N+1}(a)T_N(x)$ which vanishes at $x = a = \pm 1$
- THE GAUSS–LOBATTO–CHEBYSHEV FORMULA (exact for $u \in \mathbb{P}_{2N}$)

$$\int_{-1}^{1} u(x)\omega(x)\,dx = \frac{\pi}{2N+1}\left[u(\tilde{\xi}_0) + u(\tilde{\xi}_N) + 2\sum_{j=1}^{N-1} u(\tilde{\xi}_j)\right],$$

with $\tilde{\xi}_j = \cos\left(\frac{j\pi}{N}\right)$ (the Gauss–Lobatto points located in the interior of the domain and on both boundaries)
Proof via application of the theorem quoted above.

Chebyshev Polynomials — Numerical Integration Formulae (III)

- The Gauss–Lobatto–Chebyshev collocation points are most commonly used in Chebyshev spectral methods, because this set of points also includes the boundary points (which makes it possible to easily incorporate the boundary conditions in the collocation approach)
- Using the Gauss–Lobatto–Chebyshev points, the orthogonality relation for the Chebyshev polynomials T_k and T_l with $0 \le k, l \le N$ can be written as

$$(T_k, T_l)_\omega = \int_{-1}^{1} T_k T_l \omega \, dx = \frac{\pi}{N} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} T_k(\tilde{\xi}_j) T_l(\tilde{\xi}_j) = \frac{\pi \bar{c}_k}{2} \delta_{kl},$$

where

$$\bar{c}_k = \begin{cases} 2 & \text{if } k = 0, \\ 1 & \text{if } 1 \le k \le N-1, \\ 2 & \text{if } k = N \end{cases}$$

- Note similarity to the corresponding discrete orthogonality relation obtained for the trigonometric polynomials

Chebyshev Approximation — Galerkin Approach (I)

- Consider an approximation of $u \in L^2_\omega(I)$ in terms of a truncated Chebyshev series $u_n(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$
- Cancel the projections of the residual $R_N = u - u_N$ on the $N+1$ first basis function (i.e., the Chebyshev polynomials)

$$(R_N, T_l)_\omega = \int_{-1}^{1} \left(u T_l \omega - \sum_{k=0}^{N} \hat{u}_k T_k T_l \omega \right) dx = 0, \quad l = 0, \dots, N$$

- Taking into account the orthogonality condition, expressions for the Chebyshev expansions coefficients are obtained

$$\hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^{1} u T_k \omega \, dx,$$

which can be evaluated using, e.g., the Gauss–Lobatto–Chebyshev quadratures .

- Question — What happens on the boundary?

Chebyshev Approximation — Galerkin Approach (II)

- Let $P_N : L^2_\omega(I) \to \mathbb{P}_N$ be the orthogonal projection on the subspace $\mathbb{P}_N$ of polynomials of degree $\le N$
- *THEOREM* — For all μ and σ such that $0 \le \mu \le \sigma$, there exists a constant C such that

$$\|u - P_N u\|_{\mu,\omega} < C N^{e(\mu,\sigma)} \|u\|_{\sigma,\omega}$$

where

$$e(\mu,\sigma) = \begin{cases} 2\mu - \sigma - \frac{1}{2} & \text{for } \mu > 1, \\ \frac{3}{2}\mu - \sigma & \text{for } 0 \le \mu \le 1 \end{cases}$$

Philosophy of the proof:

1. First establish continuity of the mapping $u \to \tilde{u}$, where $\tilde{u}(\theta) = u(\cos(\theta))$, from the weighted Sobolev space $H^m_\omega(I)$ into the corresponding periodic Sobolev space $H^m_p(-\pi,\pi)$
2. Then leverage analogous approximation error bounds established for the case of trigonometric basis functions

Chebyshev Approximation — Collocation Approach (I)

- Consider an approximation of $u \in L^2_\omega(I)$ in terms of a truncated Chebyshev series (expansion coefficients as the unknowns) $u_N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$
- Cancel the residual $R_N = u - u_N$ on the set of Gauss–Lobatto–Chebyshev collocation points x_j, $j = 0, \dots, N$ (one could choose other sets of collocation points as well)

$$u(x_j) = \sum_{k=0}^{N} \hat{u}_k T_k(x_j), \quad j = 0, \dots, N$$

- Noting that $T_k(x_j) = \cos\left(k \cos^{-1}(\cos(\frac{j\pi}{N}))\right) = \cos(k\frac{j\pi}{N})$ and denoting $u_j \triangleq u(x_j)$ we obtain

$$u_j = \sum_{k=0}^{N} \hat{u}_k \cos\left(k \frac{\pi j}{N}\right), \quad j = 0, \dots, N$$

- The above system of equations can be written as $U = \mathcal{T}\hat{U}$, where U and $\hat{U}$ are vectors of grid values and expansion coefficients, respectively.

CHEBYSHEV APPROXIMATION — COLLOCATION APPROACH (II)

- In fact, the matrix $\mathcal{T}$ is invertible and

$$[\mathcal{T}^{-1}]_{jk} = \frac{2}{\bar{c}_j \bar{c}_k N} \cos\left(\frac{k\pi j}{N}\right), \quad j,k = 0,\dots,N$$

- Consequently, the expansion coefficients can be expressed as follows

$$\hat{u}_k = \frac{2}{\bar{c}_k N} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} u_j \cos\left(\frac{k\pi j}{N}\right) = \frac{2}{\bar{c}_k N} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} u_j \Re\left[e^{i\left(\frac{k\pi j}{N}\right)}\right], \quad k = 0,\dots,N$$

  Note that this expression is nothing else than the COSINE TRANSFORMS of U which can be very efficiently evaluated using a COSINE FFT

- The same expression can be obtained by
  - multiplying each side of $u_j = \sum_{k=0}^{N} \hat{u}_k T_k(x_j)$ by $\frac{T_l(x_j)}{\bar{c}_j}$
  - summing the resulting expression from $j = 0$ to $j = N$
  - using the DISCRETE ORTHOGONALITY RELATION $\frac{\pi}{N} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} T_k(\tilde{\xi}_j) T_l(\tilde{\xi}_j) = \frac{\pi \bar{c}_k}{2} \delta_{kl}$

CHEBYSHEV APPROXIMATION — COLLOCATION APPROACH (III)

- Note that the expression for the DISCRETE CHEBYSHEV TRANSFORM

$$\hat{u}_k = \frac{2}{\bar{c}_k N} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} u_j \cos\left(\frac{k\pi j}{N}\right), \quad k = 0,\dots,N$$

  can also be obtained by using the Gauss–Lobatto–Chebyshev quadrature to approximate the continuous expressions

$$\hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^{1} u T_k \omega \, dx, \quad k = 0,\dots,N,$$

  Such an approximation is EXACT for $u \in \mathbb{P}_N$

- Analogous expressions for the Discrete Chebyshev Transforms can be derived for other set of collocation points (Gauss, Gauss–Radau)
- Note similarities with respect to the case periodic functions and the Discrete Fourier Transform

CHEBYSHEV APPROXIMATION — COLLOCATION APPROACH (IV)

- As was the case with Fourier spectral methods, there is a very close connection between COLLOCATION–BASED INTERPOLATION and GALERKIN APPROXIMATION
- DISCRETE CHEBYSHEV TRANSFORM can be associated with an INTERPOLATION OPERATOR $P_C : C^0(I) \to \mathbb{R}^N$ defined such that $(P_C u)(x_j) = u(x_j)$, $j = 0,\dots,N$ (where x_j are the Gauss–Lobatto collocation points)
- *THEOREM* — Let $s > \frac{1}{2}$ and σ be given and $0 \le \sigma \le s$. There exists a constant C such that

$$\|u - P_C u\|_{\sigma,\omega} < C N^{2\sigma - s} \|u\|_{s,\omega}$$

  for all $u \in H^s_\omega(I)$.
  Philosophy of the proof — changing the variables to $\tilde{u}(\theta) = u(\cos(\theta))$ we convert this problem to a problem already analyzed in the context of the Fourier interpolation for periodic functions

CHEBYSHEV APPROXIMATION — COLLOCATION APPROACH (V)

- Relation between the GALERKIN and COLLOCATION coefficients, i.e.,

$$\hat{u}_k^e = \frac{2}{\pi c_k} \int_{-1}^{1} u(x) T_k(x) \omega(x) \, dx, \qquad k = 0,\dots,N$$

$$\hat{u}_k^c = \frac{2}{\bar{c}_k N} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} u_j \cos\left(\frac{k\pi j}{N}\right), \qquad k = 0,\dots,N$$

- Using the representation $u(x) = \sum_{l=0}^{\infty} \hat{u}_l^e T_l(x)$ in the latter expression and invoking the discrete orthogonality relation we obtain

$$\hat{u}_k^c = \frac{2}{\bar{c}_k N} \sum_{l=0}^{N} \hat{u}_k^e \left[\sum_{j=0}^{N} \frac{1}{\bar{c}_j} T_k(x_j) T_l(x_j)\right] + \frac{2}{\bar{c}_k N} \sum_{l=N+1}^{\infty} \hat{u}_k^e \left[\sum_{j=0}^{N} \frac{1}{\bar{c}_j} T_k(x_j) T_l(x_j)\right],$$

$$= \hat{u}_k^e \qquad + \frac{2}{\bar{c}_k N} \sum_{l=N+1}^{\infty} \hat{u}_k^e C_{kl}$$

  where

$$C_{kl} = \sum_{j=0}^{N} \frac{1}{\bar{c}_j} T_k(x_j) T_l(x_j) = \sum_{j=0}^{N} \frac{1}{\bar{c}_j} \cos\left(\frac{ki\pi}{N}\right) \cos\left(\frac{li\pi}{N}\right) = \frac{1}{2} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} \left[\cos\left(\frac{k-l}{N} i\pi\right) + \cos\left(\frac{k+l}{N} i\pi\right)\right]$$

Chebyshev Approximation — Collocation Approach (VI)

- Using the identity

$$\sum_{j=0}^{N} \cos\left(\frac{pi\pi}{N}\right) = \begin{cases} N+1, & \text{if } p = 2mN,\ m = 0, \pm1, \pm2, \dots \\ \frac{1}{2}[1+(-1)^p] & \text{otherwise} \end{cases}$$

we can calculate C_{kl} which allows us to express the relation between the Galerkin and collocation coefficients as follows

$$\hat{u}_k^c = \hat{u}_k^e + \frac{1}{\bar{c}_k}\left[\sum_{\substack{m=1 \\ 2mN>N-k}}^{\infty} \hat{u}^e_{k+2mN} + \sum_{\substack{m=1 \\ 2mN>N+k}}^{\infty} \hat{u}^e_{-k+2mN}\right]$$

- The terms in square brackets represent the ALIASING ERRORS . Their origin is precisely the same as in the Fourier (pseudo)–spectral method.
- Aliasing errors can be removed using the $3/2$ APPROACH in the same way as in the Fourier (pseudo)–spectral method

Chebyshev Approximation — Reciprocal Relations

- expressing the first N Chebyshev polynomials as functions of x^k, $k = 1, \dots, N$

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1 \end{aligned}$$

which can be written as $V = \mathbb{K}X$, where $[V]_k = T_k(x)$, $[X]_k = x^k$, and $\mathbb{K}$ is a LOWER–TRIANGULAR matrix

- Solving this system (trivially!) results in the following RECIPROCAL RELATIONS

$$\begin{aligned} 1 &= T_0(x), \\ x &= T_1(x), \\ x^2 &= \frac{1}{2}[T_0(x) + T_2(x)], \\ x^3 &= \frac{1}{4}[3T_1(x) + T_3(x)], \\ x^4 &= \frac{1}{8}[3T_0(x) + 4T_2(x) + T_4(x)] \end{aligned}$$

Chebyshev Approximation — Economization of Power Series

- Find the best polynomial approximation of order 3 of $f(x) = e^x$ on $[-1,1]$
- Construct the (Maclaurin) expansion

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$$

- Rewrite the expansion in terms of CHEBYSHEV POLYNOMIALS using the reciprocal relations

$$e^x = \frac{81}{64}T_0(x) + \frac{9}{8}T_1(x) + \frac{13}{48}T_2(x) + \frac{1}{24}T_3(x) + \frac{1}{192}T_4(x) + \dots$$

- Truncate this expansion to the 3^{rd} order and translate the expansion back to the x^k representation
- Truncation error is given by the magnitude of the first truncate term; Note that the CHEBYSHEV EXPANSION COEFFICIENTS are much smaller than the corresponding TAYLOR EXPANSION COEFFICIENTS !
- How is it possible — the same number of expansion terms, but higher accuracy?

Chebyshev Approximation — Spectral Differentiation (I)

- Assume the function approximation in the form $u_N(x) = \sum_{k=0}^N \hat{u}_k T_k(x)$
- First, note that CHEBYSHEV PROJECTION and DIFFERENTIATION do not commute, i.e., $P_N(\frac{du}{dx}) \neq \frac{d}{dx}(P_N u)$
- Sequentially applying the recurrence relation $2T_k = \frac{T'_{k+1}}{k+1} - \frac{T'_{k-1}}{k-1}$ we obtain

$$T_k'(x) = 2k\sum_{p=0}^{K} \frac{1}{c_{k-1-2p}} T_{k-1-2p}(x), \quad \text{where } K = \left[\frac{k-1}{2}\right]$$

- Consider the first derivative

$$u_N'(x) = \sum_{k=0}^{N} \hat{u}_k T_k'(x) = \sum_{k=0}^{N} \hat{u}_k^{(1)} T_k(x)$$

where, using the above expression for $T_k'(x)$, we obtain the expansion coefficients as

$$\hat{u}_k^{(1)} = \frac{2}{c_k} \sum_{\substack{p=k+1 \\ (p+k)\ \text{odd}}}^{N} p\hat{u}_p, \quad k = 0, \dots, N-1$$

and $\hat{u}_N^{(1)} = 0$

Chebyshev Approximation — Spectral Differentiation (II)

- Spectral differentiation (with the expansion coefficients as unknowns) can thus be written as
$$\hat{U}^{(1)} = \hat{\mathbb{D}}\hat{U},$$
where $\hat{U} = [\hat{u}_0 \dots, \hat{u}_N]^T$, $\hat{U}^{(1)} = [\hat{u}_0^{(1)} \dots, \hat{u}_N^{(1)}]^T$, and $\hat{\mathbb{D}}$ is an UPPER–TRIANGULAR matrix with entries deduced based on the previous expression
- For the second derivative one obtains similarly
$$u_N''(x) = \sum_{k=0}^{N} \hat{u}_k^{(2)} T_k(x)$$
$$\hat{u}_k^{(2)} = \frac{1}{c_k} \sum_{\substack{p=k+2 \\ (p+k)\,\text{even}}}^{N} p(p^2-k^2)\hat{u}_p, \quad k = 0,\dots,N-2$$
and $\hat{u}_N^{(2)} = \hat{u}_{N-1}^{(2)} = 0$
- QUESTION — What is the structure of the second–order differentiation matrix?

Chebyshev Approximation — Differentiation in Real Space (I)

- Assume the function $u(x)$ is approximated in terms of its nodal values, i.e.,
$$u(x) \cong u_N(x) = \sum_{j=0}^{N} u(x_j) C_j(x),$$
where $\{x_j\}$ are the GAUSS–LOBATTO–CHEBYSHEV points and $C_j(x)$ are the associated CARDINAL FUNCTIONS
$$C_j(x) = (-1)^{j+1} \frac{(1-x^2)}{c_j N^2 (x-x_j)} \frac{dT_N(x)}{dx} = \frac{2}{N p_j} \sum_{m=0}^{N} \frac{1}{p_m} T_m(x_j) T_m(x),$$
where
$$p_j = \begin{cases} 2 & \text{for } j = 0, N, \\ 1 & \text{for } j = 1, \dots, N-1 \end{cases}, \qquad c_j = \begin{cases} 2 & \text{for } j = N, \\ 1 & \text{for } j = 0, \dots, N-1 \end{cases}$$
The DIFFERENTIATION MATRIX $\mathbb{D}^{(p)}$ relating the nodal values of the p–th derivative $u_N^{(p)}$ to the nodal values of u is obtained by differentiating the cardinal function appropriate number of times
$$u_N^{(p)}(x_j) = \sum_{k=0}^{N} \frac{d^{(p)} C_k(x_j)}{dx^{(p)}} u(x_k) = \sum_{k=0}^{N} d_{jk}^{(p)} u(x_k), \quad j = 0, \dots, N$$

Chebyshev Approximation — Differentiation in Real Space (II)

- Expressions for the entries of the DIFFERENTIATION MATRIX $d_{jk}^{(1)}$ at the the GAUSS–LOBATTO–CHEBYSHEV collocation points
$$d_{jk}^{(1)} = \frac{\bar{c}_j}{\bar{c}_k} \frac{(-1)^{j+k}}{x_j - x_k}, \qquad 0 \le j,k \le N,\ j \ne k,$$
$$d_{jj}^{(1)} = -\frac{x_j}{2(1-x_j^2)}, \qquad 1 \le j \le N-1,$$
$$d_{00}^{(1)} = -d_{NN}^{(1)} = \frac{2N^2+1}{6},$$
- Thus in the matrix (operator) notation
$$U^{(1)} = \mathbb{D}U$$
- Note that ROWS of the differentiation matrix $\mathbb{D}$ are in fact equivalent to N–th order asymmetric finite–difference formulas on a nonuniform grid; in other words, spectral differentiation using nodal values as unknowns is equivalent to finite differences employing ALL N GRID POINTS AVAILABLE

Chebyshev Approximation — Differentiation in Physical Space (III)

- Expressions for the entries of SECOND–ORDER DIFFERENTIATION MATRIX $d_{jk}^{(2)}$ at the the GAUSS–LOBATTO–CHEBYSHEV collocation points ($U^{(2)} = \mathbb{D}^{(2)}U$)
$$d_{jk}^{(2)} = \frac{(-1)^{j+k}}{\bar{c}_k} \frac{x_j^2 + x_j x_k - 2}{(1-x_j^2)(x_j - x_k)^2}, \qquad 1 \le j \le N-1,\ 0 \le k \le N,\ j \ne k$$
$$d_{jj}^{(2)} = -\frac{(N^2-1)(1-x_j^2)+3}{3(1-x_j^2)^2}, \qquad 1 \le j \le N-1,$$
$$d_{0k}^{(2)} = \frac{2}{3} \frac{(-1)^k}{\bar{c}_k} \frac{(2N^2+1)(1-x_k)-6}{(1-x_k)^2}, \qquad 1 \le k \le N$$
$$d_{Nk}^{(2)} = \frac{2}{3} \frac{(-1)^{N+k}}{\bar{c}_k} \frac{(2N^2+1)(1+x_k)-6}{(1+x_k)^2} \qquad ,0 \le k \le N-1$$
$$d_{00}^{(2)} = d_{NN}^{(2)} = \frac{N^4-1}{15},$$
- Note that $\qquad d_{jk}^{(2)} = \sum_{p=0}^{N} d_{jp}^{(1)} d_{pk}^{(1)}$
- Interestingly, $\mathbb{D}^2$ is not a SYMMETRIC MATRIX ...

GALERKIN APPROACH — BCS VIA BASIS RECOMBINATION

- Consider an ELLIPTIC BOUNDARY VALUE PROBLEM (BVP) :

$$
\begin{aligned}
-\nu u'' + au' + bu &= f, && \text{in } [-1,1]\\
\alpha_- u + \beta_- u' &= g_- && x=-1\\
\alpha_+ u + \beta_+ u' &= g_+ && x=1
\end{aligned}
$$

- Chebyshev polynomials do not satisfy homogeneous boundary conditions, hence standard Galerkin approach is not directly applicable.
- BASIS RECOMBINATION :
  - Convert the BVP to the corresponding form with HOMOGENEOUS BOUNDARY CONDITIONS (cf. page 72)
  - Take linear combinations of Chebyshev polynomials to construct a new basis satisfying HOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS $\varphi_k(\pm 1) = 0$

$$
\varphi_k(x) = \begin{cases} T_k(x) - T_0(x) = T_k - 1, & k - \text{even}\\ T_k(x) - T_1(x), & k - \text{odd} \end{cases}
$$

  Note that the new basis preserves orthogonality

GALERKIN APPROACH — BCS VIA TAU APPROACH (I)

- THE TAU METHOD (Lanczos, 1938) consists in using a Galerkin approach in which explicit enforcement of the boundary conditions replaces projections on some of the test functions
- Consider the residual

$$
R_N(x) = -\nu u_N'' + a u_N' + b u_N - f,
$$

  where $u_N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$

- Cancel projections of the residual on the first $N-2$ basis functions

$$
(R_N, T_l)_\omega = \sum_{k=0}^{N} \left(-\nu \hat{u}_k^{(2)} + a\hat{u}_k^{(1)} + b\hat{u}_k \right) \int_{-1}^{1} T_k T_l \omega \, dx - \int_{-1}^{1} f T_l \omega \, dx, \quad l = 0, \dots, N-2
$$

- Thus, using orthogonality, we obtain

$$
-\nu \hat{u}_k^{(2)} + a\hat{u}_k^{(1)} + b\hat{u}_k = \hat{f}_k, \quad k = 0, \dots, N-2
$$

  where $\hat{f}_k = \int_{-1}^{1} f T_k \omega \, dx$

GALERKIN APPROACH — BCS VIA TAU APPROACH (II)

- Noting that $T_k(\pm 1) = (\pm 1)^k$ and $T_k'(\pm 1) = (\pm 1)^{k+1} k^2$, the BOUNDARY CONDITIONS are enforced by supplementing the residual equations with

$$
\sum_{k=0}^{N} (-1)^k (\alpha_- - \beta_- k^2) \hat{u}_k = g_-
$$

$$
\sum_{k=0}^{N} (-1)^k (\alpha_+ + \beta_+ k^2) \hat{u}_k = g_+
$$

- Expressing $\hat{u}_k^{(2)}$ and $\hat{u}_k^{(1)}$ in terms of $\hat{u}_k$ via the Chebyshev spectral differentiation matrices we obtain the following system

$$
\mathbb{A}\hat{U} = \hat{F}
$$

  where $\hat{U} = [\hat{u}_0, \dots, \hat{u}_N]^T$, $F = [\hat{f}_0, \dots, \hat{f}_{N-2}, g_-, g_+]$ and the matrix $\mathbb{A}$ is obtained by adding the two rows representing the boundary conditions (see above) to the matrix $\mathbb{A}_1 = -\nu\hat{\mathbb{D}}^2 + a\hat{\mathbb{D}} + bI$.

- When the domain boundary is not just a point (e.g., in 2D / 3D), formulation of the Tau method becomes somewhat more involved

COLLOCATION METHOD (I)

- Consider the residual

$$
R_N(x) = -\nu u_N'' + a u_N' + b u_N - f,
$$

  where $u_N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$

- Cancel this residual at $N-1$ GAUSS–LOBATTO–CHEBYSHEV collocation points located in the interior of the domain

$$
-\nu u_N''(x_j) + a u_N'(x_j) + b u_N(x_j) = f(x_j), \quad j = 1, \dots, N-1
$$

- Enforce the two boundary conditions at endpoints

$$
\alpha_- u_N(x_N) + \beta_- u_N'(x_N) = g_-
$$

$$
\alpha_+ u_N(x_0) + \beta_+ u_N'(x_0) = g_-
$$

  Note that this shows the utility of using the GAUSS–LOBATTO–CHEBYSHEV collocation points

Collocation Method (II)

- Consequently, the following system of $N+1$ equations is obtained

$$\sum_{k=0}^{N}(-\nu d_{jk}^{(2)}+ad_{jk}^{(1)})u_N(x_j)+bu_N(x_j)=f(x_j),\quad j=1,\dots,N-1$$

$$\alpha_- u_N(x_N)+\beta_-\sum_{k=0}^{N}d_{Nk}^{(1)}u_N(x_k)=g_-$$

$$\alpha_+ u_N(x_0)+\beta_+\sum_{k=0}^{N}d_{0k}^{(1)}u_N(x_k)=g_+$$

  which can be written as $\mathbb{A}_c U=F$, where $[\mathbb{A}_c]_{jk}=[\mathbb{A}_{c0}]_{jk}$, $j,k=1,\dots,N-1$ with $\mathbb{A}_{c0}$ given by

$$\mathbb{A}_{c0}=(-\nu\mathbb{D}^2+a\mathbb{D}+b\mathbb{I})U$$

  and the BOUNDARY CONDITIONS above added as the rows 0 and N of $\mathbb{A}_c$

- Note that the matrix corresponding to this system of equations may be POORLY CONDITIONED , so special care must be exercised when solving this system for large N.
- Similar approach can be used when the nodal values $u(x_j)$, rather than the Chebyshev coefficients $\hat{u}_k$ are unknowns

Chebyshev Methods — Nonconstant Coefficients and Nonlinear Equations

- When the equations has NONCONSTANT COEFFICIENTS , similar difficulties as in the Fourier case are encountered (evaluation of CONVOLUTION SUMS)
- Consequently, the COLLOCATION (pseudo–spectral) approach is preferable along the guidelines laid out in the case of the Fourier spectral methods
- Assuming $a=a(x)$ in the elliptic boundary value problem, we need to make the following modification to $\mathbb{A}_c$:

$$\mathbb{A}'_{c0}=(-\nu\mathbb{D}^2+\mathbb{D}'+b\mathbb{I})U,$$

  where $\mathbb{D}'=[a(x_j)d_{jk}^{(1)}]$, $j,k=1,\dots,N$

- For the Burgers equation $\partial_t u+\frac{1}{2}\partial_x u^2-\nu\partial_x^2 u$ we obtain at every time step n

$$(\mathbb{I}-\Delta t\,\nu\,\mathbb{D}^{(2)})U^{n+1}=U^n-\frac{1}{2}\Delta t\,\mathbb{D}W^n,$$

  where $[W^n]_j=[U^n]_j[U^n]_j$; Note that an algebraic system has to be solved at each time step

Epilogue — Domain Decomposition

- Motivation:
  - treatment of problem in IRREGULAR DOMAINS
  - STIFF PROBLEMS
- PHILOSOPHY — partition the original domain Ω into a number of SUBDOMAINS $\{\Omega_m\}_{m=1}^{M}$ and solve the problem separately on each those while respecting consistency conditions on the interfaces
- SPECTRAL ELEMENT METHOD
  - consider a collection of problem posed on each subdomain Ω_m

$$\mathcal{L}u_m=f$$

$$u_{m-1}(a_m)=u_m(a_m),\qquad u_m(a_{m+1})=u_{m+1}(a_{m+1})$$

  - Transform each subdomain Ω_m to $I=[-1,1]$
  - use a separate set of N_m ORTHOGONAL POLYNOMIALS to approximate the solution on every subinterval
  - boundary conditions on interfaces provide coupling between problems on subdomains