

HOMEWORK #2

Due: November 6 (Tuesday) by midnight

Submit your solutions in the form of a *single* MATLAB m-file via Email to the instructor.

Please embed your narrative answers in this file using the instruction `disp(...)`, or submit them as a separate document in the PDF format (no Word files will be accepted!).

Late submissions will not be considered.

1. You are given the following two functions $f_1, f_2 : [0, 2\pi] \rightarrow \mathbb{R}$

$$\begin{aligned} f_1(x) &= e^{\sin(x)}, \\ f_2(x) &= e^{\sin(\frac{1}{2}x)}. \end{aligned}$$

Assuming that these functions are periodic on \mathbb{R} (with period 2π), approximate numerically the definite integrals $\int_0^{2\pi} f_1(x) dx$ and $\int_0^{2\pi} f_2(x) dx$ using a suitable *periodic* Gaussian quadrature. Plot the relative error (with respect to the exact values of the integrals) as a function of the number N of subintervals discretizing $[0, 2\pi]$, where $2^3 \leq N \leq 2^{16}$. What is the order of accuracy of the quadrature in the two cases? What explains the difference?

The exact value of $\int_0^{2\pi} f_1(x) dx$ can be found analytically (for example, using Maple), whereas $\int_0^{2\pi} f_2(x) dx = 12.4175160714222$.
[4 points]

2. Consider the periodic extensions (with period 2π) of the following two functions

$$\begin{aligned} g_1(x) &= \begin{cases} -1 & 0 \leq x \leq \pi \\ 1 & \pi < x \leq 2\pi \end{cases}, \\ g_2(x) &= \begin{cases} \frac{2}{\pi}x & 0 \leq x \leq \frac{\pi}{2} \\ -\frac{2}{\pi}x + 2 & \frac{\pi}{2} < x \leq \frac{3\pi}{2} \\ \frac{2}{\pi}x - 4 & \frac{3\pi}{2} < x \leq 2\pi \end{cases}. \end{aligned}$$

Construct truncated Fourier series in the form

$$g_i^N = \sum_{k=1}^N \frac{\hat{g}_{i,k}}{\sqrt{\pi}} \sin(kx), \quad i = 1, 2, \tag{1}$$

where the Fourier coefficients $\hat{g}_{i,k} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} g_i(x) \sin(kx) dx$, $k = 1, \dots, N$, can be approximated with the periodic Gaussian quadrature using $N_0 = 10^5$ grid points. Then plot

- (a) the magnitudes $|\hat{g}_{i,k}|$, $i = 1, 2$, of the Fourier coefficients as a function of k assuming that the number of terms in series (1) is $N = 100$ (one figure),
- (b) the relative error norms

$$E_2 = \frac{\|g_i^N - g_i\|_{L_2(0,2\pi)}}{\|g_i\|_{L_2(0,2\pi)}}, \quad E_\infty = \frac{\|g_i^N - g_i\|_{L_\infty(0,2\pi)}}{\|g_i\|_{L_\infty(0,2\pi)}}$$

as functions of the number of terms $N = 1, \dots, 100$ in Fourier series (1) for $i = 1, 2$; the norms $\|\cdot\|_{L_2(0,2\pi)}$ and $\|\cdot\|_{L_\infty(0,2\pi)}$ can be approximated with, respectively, the MATLAB functions `norm(vec,2)` and `norm(vec,inf)`, where `vec` represents a function discretized on a uniform grid with N_0 points (separate figure for each norm),

- (c) the functions $g_i(x)$ and the corresponding truncated Fourier series $g_i^N(x)$ for $N = 1, 21, 41, \dots$ and $i = 1, 2$ as functions of $x \in [0, 2\pi]$ (separate figure for each function).

Explain the behaviour of the errors E_2 and E_∞ with respect to the number of terms N in series (1) for the two functions g_1 and g_2 .

[6 points]

3. **[Bonus Question]** Construct and plot a numerical approximation of a periodic function with period 2π and with *minimum* regularity necessary to prevent Runge's phenomenon. Note that such function should possess square-integrable derivatives of (fractional) order $1/2 + \epsilon$, where $\epsilon > 0$.

[2 points]