MATH 745, QUIZ #1 SOLUTIONS

(1) $$\frac{d^2y}{dt^2} = -y \ , \quad y(0) = 1 \ , \quad \frac{dy}{dt}(0) = 0$$

This is a second-order ODE; need to transform it to a system of first-order ODEs.

$$\begin{cases} \dfrac{dy}{dt} = u \\[2ex] \dfrac{du}{dt} = -y \end{cases}$$

Using vector/matrix notation

$$X = \begin{bmatrix} u \\ y \end{bmatrix}, \quad \frac{d}{dt}\begin{bmatrix} u \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} u \\ y \end{bmatrix}$$

Thus, we have equivalently $\quad \dot{X} = AX, \quad X(0) = \begin{bmatrix} 0 \\ +1 \end{bmatrix}$

Matrix A has eigenvalues $\pm i$, hence system $\dot{X} = AX$ is equivalent to $\begin{cases} \dot{v}_1 = i v_1 \\ \dot{v}_2 = -i v_2 \end{cases} \Rightarrow \lambda = i\lambda_i = i$

$\lambda_i = 1, \ \lambda_r = 0$

a) Euler's explicit method

$$X^{(n+1)} = X^{(n)} + \Delta t A X^{(n)} = (I + \Delta t A) X^{(n)}$$

Method is <u>unstable</u> for all values of Δt, since the eigenvalues are purely imaginary.

b) Leap frog method

$$x^{(n+1)} = x^{(n-1)} + 2\Delta t\, A\, x^{(n)}$$

Method <u>conditionally stable</u> : $\Delta t \le \frac{1}{|\lambda_i|} = 1$

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(2)

$$\mathcal{L}y = g \Rightarrow \begin{cases} \frac{d^2y}{dx^2} - ay = g & \text{in } (-1,1) \\ y(-1) = y(0) = 0 \end{cases}$$

~~Approximate~~ Orthogonal basis functions

$$(\phi_i, \phi_j) = \int_{-1}^{1} \phi_i \phi_j \, dx = \delta_{ij}, \quad i,j = 1, \dots, N$$

Approximate solution $y_N = \sum_{i=1}^{N} \alpha_i \phi_i(x)$, $\alpha_i \in \mathbb{R}$

Residual $R_N(x) = \sum_{i=1}^{N} \alpha_i \frac{d^2 \phi_i}{dx^2} - a \sum_{i=1}^{N} \alpha_i \phi_i - g$

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a) Spectral Galerkin method

The orthogonality condition $(R_N, \phi_j) = 0, \; j=1,\dots,N$

$$\sum_{i=1}^{N} \alpha_i \int_{-1}^{1} \left(\frac{d^2\phi_i}{dx^2} - a\phi_i, \phi_j \right) dx = \int_{-1}^{1} g \phi_j \, dx, \quad j = 1, \dots, N$$

After integration by parts

$$-\sum_{i=1}^{N} \alpha_i \int_{-1}^{1} \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} + a\phi_i\phi_j \, dx = \int_{-1}^{1} f\phi_j \, dx, \quad j=1,\dots,N$$

Denoting $X = [\alpha_1, \dots, \alpha_N]^T$

$$b = \left[\int_{-1}^{1} f\phi_1 \, dx, \dots, \int_{-1}^{1} f\phi_N \, dx \right]^T$$

we obtain the algebraic problem $AX = b$

where $[A]_{ij} = -\int_{-1}^{1} \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} + a\phi_i\phi_j \, dx$

b) Collocation approach

Define $\{x_i\}_{i=1}^{N} \in (-1,1)$, such that $x_i \neq x_j$, $i \neq j$

Then

$$R_N(x_j) = 0, \; j=1,\dots,N \Rightarrow \sum_{i=1}^{N} \alpha_i \left(\frac{d^2\phi_i}{dx^2}(x_j) - a\phi_i(x_j) \right) = f(x_j)$$

Defining $g = [f(x_1), \dots, f(x_N)]^T$

we have

$$AX = g$$

where $$[A]_{ij} = \frac{d^2\phi_i}{dx^2}(x_j) - a\phi_i(x_j)$$