PART III
Review of (Abstract) Approximation Theory

Although this may seem a paradox,
all exact science is dominated by the idea of approximation.
— Bertrand Russell (1872–1970)

Agenda

- Consider a real or complex linear space V; A SCALAR PRODUCT is real or complex number (x, y) associated with the elements $x, y \in V$ with the following properties:
  - (x, x) is real, $(x, x) \geq 0$, $(x, x) = 0$ only if $x = 0$,
  - $(x, y) = \overline{(y, x)}$,
  - $(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 (x_1, y) + \alpha_2 (x_2, y)$
- A normed space V is said to be UNITARY if its norm and scalar product are connected via the following relation: $\|x\| = (x, x)^{1/2}$
- A complete unitary space H is called a HILBERT SPACE

- Two elements x and y of a Hilbert space V are said to be mutually ORTHOGONAL $(x \perp y)$ if $(x, y) = 0$. A countable set of elements $x_1, x_2, \dots, x_k, \dots$ is said to be ORTHONORMAL (or to form AN ORTHONORMAL SYSTEMS) if $(x_i, x_j) = \delta_{ij}$
- The following properties hold:
  - $x \perp 0$ for all $x \in V$
  - $x \perp x$ only if $x = 0$
  - if $x \perp \mathcal{A}$, i.e., $x \perp y$ for all $y \in \mathcal{A} \subseteq V$, then x is also orthogonal to the linear hull $\mathcal{L}(\mathcal{A})$
  - if $x \perp y_n$ $(n = 1, 2, \dots)$ and $y_n \rightarrow y$, then $x \perp y$
  - if $\mathcal{A}$ is dense in V and $x \perp \mathcal{A}$, then $x = 0$
- SCHMIDT ORTHOGONALIZATION — Let $\mathcal{A}$ be a set of countably many linearly independent elements $x_1, x_2, \dots, x_k, \dots$ of a Hilbert space H. Then there is an orthonormal system $\mathcal{F} = \{e_i \in V : (e_i, e_j) = \delta_{ij}\}$, such that the linear hulls of $\mathcal{A}_k = \{x_j : j = 1, \dots, k\}$ and $\mathcal{F}_k = \{e_j : j = 1, \dots, k\}$ are the same for all k.

- Let $\{e_1, e_2, \dots\}$ be an orthonormal system in a Hilbert space H and let H_k be the linear hull of $\{e_1, \dots, e_k\}$. Then for every $x \in H$ the element $a = \sum_{j=1}^{k} (x, e_j)\, e_j \in H_k$ has the property that $\|x - a\| \leq \|x - y\|$ for all $y \in H_k$. The numbers (x, e_j) are called THE FOURIER COEFFICIENTS relative to the orthonormal system $\{e_1, e_2, \dots\}$. Furthermore, from $\|x - a\|^2 \geq 0$ follows the BESSEL INEQUALITY :

$$\sum_{j=1}^{k} |(x, e_j)|^2 \leq (x, x)$$

- If $\mathcal{A}$ is a given set in a Hilbert space H, then

$$\mathcal{A}^{\perp} = \{x : (x, a) = 0 \text{ for all } a \in \mathcal{A}\}$$

is a closed linear subspace of H. It is, therefore, itself a Hilbert space and is called THE ORTHOGONAL COMPLEMENT OF $\mathcal{A}$

- If H_1 is a closed linear subspace of a Hilbert space H and H_2 is its orthogonal complement, then every $x \in H$ can be uniquely represented in the form

$$x = x_1 + x_2, \quad (x_1 \in H_1, x_2 \in H_2)$$

We write $H = H_1 \oplus H_2$ and call H an orthogonal sum of H_1 and H_2.

- Since

$$\|x - x_1\| = \rho(x, H_1) = \inf_{y_1 \in H_1} \{\|x - y_1\|\},$$
$$\|x - x_2\| = \rho(x, H_2) = \inf_{y_2 \in H_2} \{\|x - y_2\|\},$$

one calls x_1 and x_2 the ORTHOGONAL PROJECTIONS of x on H_1 and H_2, respectively.

- Let $\{e_1, e_2, \dots\}$ be a countable orthonormal system in a Hilbert space H. By Bessel inequality, the series $\sum_{j=1}^{\infty}(x, e_j)\, e_j = \lim_{n\to\infty} \sum_{j=1}^{n}(x, e_j)\, e_j$ defines an element of H for every $x \in H$. This is called THE FOURIER SERIES OF x
- The partial sum $s_n = \sum_{j=1}^{n}(x, e_j)\, e_j$ is the orthogonal projection of x on the subspace $H_n = \mathcal{L}(\{e_1, \dots, e_n\})$. One has $\|s_n\|^2 = \sum_{j=1}^{n} |(x, e_j)|^2$
- If the system $\{e_1, \dots, e_k, \dots\}$ is complete in H, i.e., $\overline{\mathcal{L}(\{e_1, \dots, e_k, \dots\})} = H$, then the Fourier series for any $x \in H$ converges to x

- An orthonormal system is said to be CLOSED if THE PARCEVAL EQUATION

$$\sum_{j=1}^{\infty} |(x, e_j)|^2 = \|x\|^2$$

holds for every $x \in H$. An orthonormal system is closed IFF it is complete.

- An orthonormal system in a separable Hilbert space is at most countable

- Statement of a GENERAL APPROXIMATION PROBLEM IN A HILBERT SPACE H — consider a fixed element $f \in H$ and $\mathcal{G}_n \subseteq H$ which is a finite–dimensional subspace of H (with the same norm). Want to find an element $\hat{g} \in \mathcal{G}_n$ such that

$$D(f, \mathcal{G}_n, \|\cdot\|) \triangleq \inf_{g \in \mathcal{G}_n} \{\|f - g\|\} = \|f - \hat{g}\|$$

The element $\hat{g}$ is called THE BEST APPROXIMATION and the number $D(f, \mathcal{G}_n, \|\cdot\|)$ is called THE DEFECT .

- Issues:
  - Does the best approximation $\hat{g}$ exist?
  - Can $\hat{g}$ be uniquely determined?
  - How can $\hat{g}$ be computed?

- The approximation problem in a Hilbert space H has a unique solution $\hat{g}$ for which $(\hat{g} - f, h) = 0$ holds for all $h \in \mathcal{G}_n$. If $\{e_1, \ldots, e_n\}$ is a basis of $\mathcal{G}_n$, then

$$\hat{g} = \sum_{j=1}^{n} c_j^{(n)} e_j$$

with

$$\sum_{j=1}^{n} c_j^{(n)} (e_j, e_k) = (f, e_k), \quad j = 1, \ldots, n \qquad (\bigstar)$$

and the approximation error is

$$\|f - \hat{g}\|^2 = (f - \hat{g}, f - \hat{g}) = \|f\|^2 - \sum_{j=1}^{n} c_j^{(n)} (e_j, f)$$

- Thus, the Fourier coefficients $c_j^{(n)}$, $j = 1, \ldots, n$, can be calculated by solving an algebraic system $(\bigstar)$ with the Hermitian, positive–definite matrix $A_{jk} = (e_j, e_k)$ (the so called GRAM MATRIX).
- If the basis $\{e_1, \ldots, e_n\}$ is orthogonal, the system becomes decoupled and the Fourier coefficients can be calculated simply as $c_k^{(n)} = (f, e_k)$

- Assume that c_j, $j = 1, 2, \dots$ are the Fourier coefficients related to an approximation of some function $f = \sum_{j=1}^{n} c_j e_j$

- The RATE OF CONVERGENCE of this approximation is:

  - ALGEBRAIC with order k if for $j >> 1$

$$\lim_{j\to\infty} |c_j| j^k < \infty, \quad \text{or, equivalently,} \ \ |c_j| \sim \mathcal{O}(j^{-k})$$

  - EXPONENTIAL OR SPECTRAL with index r if for ANY $k > 0$

$$\lim_{j\to\infty} |c_j| j^k < \infty, \quad \text{or, equivalently,} \ \ |c_j| \sim \mathcal{O}(\exp(-qj^r)), \quad r, q \in \mathbb{R}^+$$

    spectral convergence can be:

    - SUBGEOMETRIC when $r < 1$,
    - GEOMETRIC when $r = 1$, and
    - SUPERGEOMETRIC otherwise