Agenda

Chebyshev Approximation (I)
Galerkin Approach
Collocation Approach
Reciprocal Relations & Economization of Power Series

Chebyshev Approximation (II)
Spectral Differentiation
Differentiation in Real Space

Implementation of Boundary Conditions
Galerkin Approach & Basis Recombination
Galerkin Approach & Tau Method
Collocation Method

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▶ Consider an approximation of $u \in L^2_\omega(I)$ in terms of a TRUNCATED CHEBYSHEV SERIES $u_N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$

▶ Cancel the projections of the residual $R_N = u - u_N$ on the $N + 1$ first basis function (i.e., the Chebyshev polynomials)

$$(R_N, T_l)_\omega = \int_{-1}^{1} \left(u T_l \omega - \sum_{k=0}^{N} \hat{u}_k T_k T_l \omega \right) dx = 0, \quad l = 0, \ldots, N$$

▶ Taking into account the orthogonality condition, expressions for the Chebyshev expansions coefficients are obtained

$$\hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^{1} u T_k \omega \, dx,$$

which can be evaluated using, e.g., the GAUSS-LOBATTO-CHEBYSHEV QUADRATURES .

▶ QUESTION — What happens on the boundary?

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Theorem

Let $P_N : L^2_\omega(I) \to \mathbb{P}_N$ be the orthogonal projection on the subspace $\mathbb{P}_N$ of polynomials of degree $\leq N$. For all μ and σ such that $0 \leq \mu \leq \sigma$, there exists a constant C such that

$$\|u - P_N u\|_{\mu,\omega} < C N^{e(\mu,\sigma)} \|u\|_{\sigma,\omega}$$

where
$$e(\mu,\sigma) = \begin{cases} 2\mu - \sigma - \dfrac{1}{2} & \text{for } \mu > 1, \\ \dfrac{3}{2}\mu - \sigma & \text{for } 0 \leq \mu \leq 1 \end{cases}$$

"Philosophy" of the proof.

1. First establish continuity of the mapping $u \to \tilde{u}$, where $\tilde{u}(\theta) = u(\cos(\theta))$, from the weighted Sobolev space $H^m_\omega(I)$ into the corresponding periodic Sobolev space $H^m_p(-\pi, \pi)$

2. Then leverage analogous approximation error bounds established for the case of trigonometric basis functions

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▶ Consider an approximation of $u \in L^2_\omega(I)$ in terms of a truncated Chebyshev series (expansion coefficients as the unknowns) $u_N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$

▶ Cancel the residual $R_N = u - u_N$ on the set of GAUSS-LOBATTO-CHEBYSHEV collocation points x_j, $j = 0, \ldots, N$ (one could choose other sets of collocation points as well)

$$u(x_j) = \sum_{k=0}^{N} \hat{u}_k T_k(x_j), \ \ j = 0, \ldots, N$$

▶ Noting that $T_k(x_j) = \cos\left(k \cos^{-1}(\cos(\frac{j\pi}{N}))\right) = \cos(k\frac{j\pi}{N})$ and denoting $u_j \triangleq u(x_j)$ we obtain

$$u_j = \sum_{k=0}^{N} \hat{u}_k \cos\left(k\frac{\pi j}{N}\right), \ \ j = 0, \ldots, N$$

▶ The above system of equations can be written as $U = \mathcal{T}\hat{U}$, where U and $\hat{U}$ are vectors of grid values and expansion coefficients, respectively.

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▶ In fact, the matrix $\mathcal{T}$ is invertible and

$$[\mathcal{T}^{-1}]_{jk} = \frac{2}{\overline{c}_j \overline{c}_k N} \cos\left(\frac{k\pi j}{N}\right), \quad j, k = 0, \ldots, N$$

▶ Consequently, the expansion coefficients can be expressed as follows

$$\hat{u}_k = \frac{2}{\overline{c}_k N} \sum_{j=0}^{N} \frac{1}{\overline{c}_j} u_j \cos\left(\frac{k\pi j}{N}\right) = \frac{2}{\overline{c}_k N} \sum_{j=0}^{N} \frac{1}{\overline{c}_j} u_j \Re\left[e^{i\left(\frac{k\pi j}{N}\right)}\right], \quad k = 0, \ldots, N$$

Note that this expression is nothing else than the COSINE TRANSFORM of U which can be very efficiently evaluated using a COSINE FFT

▶ The same expression can be obtained by
  ▶ multiplying each side of $u_j = \sum_{k=0}^{N} \hat{u}_k T_k(x_j)$ by $\frac{T_l(x_j)}{\overline{c}_j}$
  ▶ summing the resulting expression from $j = 0$ to $j = N$
  ▶ using the DISCRETE ORTHOGONALITY RELATION

$$\frac{\pi}{N} \sum_{j=0}^{N} \frac{1}{\overline{c}_j} T_k(\tilde{\xi}_j) T_l(\tilde{\xi}_j) = \frac{\pi \overline{c}_k}{2} \delta_{kl}$$

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▶ Note that the expression for the DISCRETE CHEBYSHEV TRANSFORM

$$\hat{u}_k = \frac{2}{\bar{c}_k N} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} u_j \cos\left(\frac{k\pi j}{N}\right), \quad k = 0, \ldots, N$$

can also be obtained by using the Gauss-Lobatto-Chebyshev quadrature to approximate the continuous expressions

$$\hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^{1} u T_k \omega \, dx, \quad k = 0, \ldots, N,$$

Such an approximation is EXACT for $u \in \mathbb{P}_N$

▶ Analogous expressions for the Discrete Chebyshev Transforms can be derived for other set of collocation points (Gauss, Gauss-Radau)

▶ Note similarities with respect to the case periodic functions and the Discrete Fourier Transform

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▶ As was the case with Fourier spectral methods, there is a very close connection between COLLOCATION-BASED INTERPOLATION and GALERKIN APPROXIMATION

▶ DISCRETE CHEBYSHEV TRANSFORM can be associated with an INTERPOLATION OPERATOR $P_C : C^0(I) \to \mathbb{R}^N$ defined such that $(P_C u)(x_j) = u(x_j)$, $j = 0, \dots, N$ (where x_j are the Gauss-Lobatto collocation points)

Theorem

Let $s > \frac{1}{2}$ and σ be given and $0 \le \sigma \le s$. There exists a constant C such that

$$\|u - P_C u\|_{\sigma,\omega} < CN^{2\sigma - s} \|u\|_{s,\omega}$$

for all $u \in H_\omega^s(I)$.

Outline of the Proof.

Changing the variables to $\tilde{u}(\theta) = u(\cos(\theta))$ we convert this problem to a problem already analyzed in the context of the Fourier interpolation for periodic functions   □

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▶ Relation between the GALERKIN and COLLOCATION coefficients, i.e.,

$$\hat{u}_k^e = \frac{2}{\pi c_k} \int_{-1}^{1} u(x) \, T_k(x) \omega(x) \, dx, \qquad k = 0, \dots, N$$

$$\hat{u}_k^c = \frac{2}{\bar{c}_k N} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} u_j \cos\left(\frac{k\pi j}{N}\right), \qquad k = 0, \dots, N$$

▶ Using the representation $u(x) = \sum_{l=0}^{\infty} \hat{u}_l^e T_l(x)$ in the latter expression and invoking the discrete orthogonality relation we obtain

$$\hat{u}_k^c = \frac{2}{\bar{c}_k N} \sum_{l=0}^{N} \hat{u}_l^e \left[\sum_{j=0}^{N} \frac{1}{\bar{c}_j} T_k(x_j) T_l(x_j) \right] + \frac{2}{\bar{c}_k N} \sum_{l=N+1}^{\infty} \hat{u}_l^e \left[\sum_{j=0}^{N} \frac{1}{\bar{c}_j} T_k(x_j) T_l(x_j) \right],$$

$$= \hat{u}_k^e \qquad\qquad\qquad + \frac{2}{\bar{c}_k N} \sum_{l=N+1}^{\infty} \hat{u}_l^e \, C_{kl}$$

$$\text{where} \quad C_{kl} = \sum_{j=0}^{N} \frac{1}{\bar{c}_j} T_k(x_j) T_l(x_j) = \sum_{j=0}^{N} \frac{1}{\bar{c}_j} \cos\left(\frac{kj\pi}{N}\right) \cos\left(\frac{lj\pi}{N}\right)$$

$$= \frac{1}{2} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} \left[\cos\left(\frac{k-l}{N} j\pi\right) + \cos\left(\frac{k+l}{N} j\pi\right) \right]$$

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▶ Using the identity

$$\sum_{j=0}^{N} \cos\left(\frac{pi\pi}{N}\right) = \begin{cases} N+1, & \text{if } p = 2mN, \ m = 0, \pm 1, \pm 2, \dots \\ \frac{1}{2}[1 + (-1)^p] & \text{otherwise} \end{cases}$$

we can calculate C_{kl} which allows us to express the relation between the Galerkin and collocation coefficients as follows

$$\hat{u}_k^c = \hat{u}_k^e + \frac{1}{\bar{c}_k}\left[\sum_{\substack{m=1 \\ 2mN>N-k}}^{\infty} \hat{u}_{k+2mN}^e + \sum_{\substack{m=1 \\ 2mN>N+k}}^{\infty} \hat{u}_{-k+2mN}^e\right]$$

▶ The terms in square brackets represent the ALIASING ERRORS . Their origin is precisely the same as in the Fourier (pseudo)-spectral method.

▶ Aliasing errors can be removed using the 3/2 APPROACH in the same way as in the Fourier (pseudo)-spectral method

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▶ expressing the first N Chebyshev polynomials as functions of x^k,
$k = 1, \ldots, N$

$$T_0(x) = 1,$$
$$T_1(x) = x,$$
$$T_2(x) = 2x^2 - 1,$$
$$T_3(x) = 4x^3 - 3x,$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$

which can be written as $V = \mathbb{K}X$, where $[V]_k = T_k(x)$, $[X]_k = x^k$, and $\mathbb{K}$ is a LOWER-TRIANGULAR matrix

▶ Solving this system (trivially!) results in the following RECIPROCAL RELATIONS

$$1 = T_0(x),$$
$$x = T_1(x),$$
$$x^2 = \frac{1}{2}[T_0(x) + T_2(x)],$$
$$x^3 = \frac{1}{4}[3T_1(x) + T_3(x)],$$
$$x^4 = \frac{1}{8}[3T_0(x) + 4T_2(x) + T_4(x)]$$

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▶ Find the best polynomial approximation of order 3 of $f(x) = e^x$ on $[-1, 1]$

▶ Construct the (Maclaurin) expansion

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$$

▶ Rewrite the expansion in terms of CHEBYSHEV POLYNOMIALS using the reciprocal relations

$$e^x = \frac{81}{64}T_0(x) + \frac{9}{8}T_1(x) + \frac{13}{48}T_2(x) + \frac{1}{24}T_3(x) + \frac{1}{192}T_4(x) + \dots$$

▶ Truncate this expansion to the 3rd order and translate the expansion back to the x^k representation

▶ Truncation error is given by the magnitude of the first truncated term; Note that the CHEBYSHEV EXPANSION COEFFICIENTS are much smaller than the corresponding TAYLOR EXPANSION COEFFICIENTS !

▶ How is it possible – the same number of expansion terms, but higher accuracy?

► Assume function approximation in the form $u_N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$

► First, note that CHEBYSHEV PROJECTION and DIFFERENTIATION do not commute, i.e., $P_N(\frac{du}{dx}) \neq \frac{d}{dx}(P_N u)$

► Sequentially applying the recurrence relation $2T_k = \frac{T'_{k+1}}{k+1} - \frac{T'_{k-1}}{k-1}$ we obtain

$$T'_k(x) = 2k \sum_{p=0}^{K} \frac{1}{c_{k-1-2p}} T_{k-1-2p}(x), \quad \text{where } K = \left[\frac{k-1}{2}\right]$$

► Consider the first derivative

$$u'_N(x) = \sum_{k=0}^{N} \hat{u}_k T'_k(x) = \sum_{k=0}^{N} \hat{u}_k^{(1)} T_k(x)$$

where, using the above expression for $T'_k(x)$, we obtain the expansion coefficients as

$$\hat{u}_k^{(1)} = \frac{2}{c_k} \sum_{\substack{p=k+1 \\ (p+k)\,\text{odd}}}^{N} p\,\hat{u}_p, \quad k = 0, \ldots, N-1, \quad \text{and} \quad \hat{u}_N^{(1)} = 0$$

▶ Spectral differentiation (with the expansion coefficients as unknowns) can thus be written as

$$\hat{U}^{(1)} = \hat{\mathbb{D}}\hat{U},$$

where $\hat{U} = [\hat{u}_0 \ldots, \hat{u}_N]^T$, $\hat{U}^{(1)} = [\hat{u}_0^{(1)} \ldots, \hat{u}_N^{(1)}]^T$, and $\hat{\mathbb{D}}$ is an UPPER-TRIANGULAR matrix with entries deduced based on the previous expression

▶ For the second derivative one obtains similarly

$$u_N''(x) = \sum_{k=0}^{N} \hat{u}_k^{(2)} T_k(x)$$

$$\hat{u}_k^{(2)} = \frac{1}{c_k} \sum_{\substack{p=k+2 \\ (p+k)\,\text{even}}}^{N} p(p^2 - k^2)\hat{u}_p, \quad k = 0, \ldots, N-2$$

and $\hat{u}_N^{(2)} = \hat{u}_{N-1}^{(2)} = 0$

▶ QUESTION — What is the structure of the second-order differentiation matrix?

▶ Assume the function $u(x)$ is approximated in terms of its nodal values, i.e.,

$$u(x) \cong u_N(x) = \sum_{j=0}^{N} u(x_j) C_j(x),$$

where $\{x_j\}$ are the GAUSS-LOBATTO-CHEBYSHEV points and $C_j(x)$ are the associated CARDINAL FUNCTIONS

$$C_j(x) = (-1)^{j+1} \frac{(1-x^2)}{c_j N^2 (x - x_j)} \frac{dT_N(x)}{dx} = \frac{2}{N p_j} \sum_{m=0}^{N} \frac{1}{p_m} T_m(x_j) T_m(x),$$

where

$$p_j = \begin{cases} 2 & \text{for } j = 0, N, \\ 1 & \text{for } j = 1, \ldots, N-1 \end{cases}, \qquad c_j = \begin{cases} 2 & \text{for } j = N, \\ 1 & \text{for } j = 0, \ldots, N-1 \end{cases}$$

▶ The DIFFERENTIATION MATRIX $\mathbb{D}^{(p)}$ relating the nodal values of the p-th derivative $u_N^{(p)}$ to the nodal values of u is obtained by differentiating the cardinal function appropriate number of times

$$u_N^{(p)}(x_j) = \sum_{k=0}^{N} \frac{d^{(p)} C_k(x_j)}{dx^{(p)}} u(x_k) = \sum_{k=0}^{N} d_{jk}^{(p)} u(x_k), \quad j = 0, \ldots, N$$

▶ Expressions for the entries of the DIFFERENTIATION MATRIX $d_{jk}^{(1)}$ at the the GAUSS-LOBATTO-CHEBYSHEV collocation points

$$d_{jk}^{(1)} = \frac{\overline{c}_j}{\overline{c}_k} \frac{(-1)^{j+k}}{x_j - x_k}, \qquad 0 \leq j, k \leq N, \; j \neq k,$$

$$d_{jj}^{(1)} = -\frac{x_j}{2(1 - x_j^2)}, \qquad 1 \leq j \leq N - 1,$$

$$d_{00}^{(1)} = -d_{NN}^{(1)} = \frac{2N^2 + 1}{6},$$

▶ Thus in the matrix (operator) notation
$$U^{(1)} = \mathbb{D} U$$

▶ Note that ROWS of the differentiation matrix $\mathbb{D}$ are in fact equivalent to N-th order asymmetric finite-difference formulas on a nonuniform grid; in other words, spectral differentiation using nodal values as unknowns is equivalent to finite differences employing ALL N GRID POINTS AVAILABLE

▶ Expressions for the entries of SECOND-ORDER DIFFERENTIATION MATRIX $d_{jk}^{(2)}$ at the the GAUSS-LOBATTO-CHEBYSHEV collocation points ($U^{(2)} = \mathbb{D}^{(2)} U$)

$$d_{jk}^{(2)} = \frac{(-1)^{j+k}}{\bar{c}_k} \frac{x_j^2 + x_j x_k - 2}{(1 - x_j^2)(x_j - x_k)^2}, \qquad 1 \le j \le N-1, \ 0 \le k \le N, \ j \ne k$$

$$d_{jj}^{(2)} = -\frac{(N^2 - 1)(1 - x_j^2) + 3}{3(1 - x_j^2)^2}, \qquad 1 \le j \le N-1,$$

$$d_{0k}^{(2)} = \frac{2}{3} \frac{(-1)^k}{\bar{c}_k} \frac{(2N^2 + 1)(1 - x_k) - 6}{(1 - x_k)^2}, \qquad 1 \le k \le N$$

$$d_{Nk}^{(2)} = \frac{2}{3} \frac{(-1)^{N+k}}{\bar{c}_k} \frac{(2N^2 + 1)(1 + x_k) - 6}{(1 + x_k)^2} \qquad ,0 \le k \le N-1$$

$$d_{00}^{(2)} = d_{NN}^{(2)} = \frac{N^4 - 1}{15},$$

▶ Note that $\qquad d_{jk}^{(2)} = \sum_{p=0}^{N} d_{jp}^{(1)} d_{pk}^{(1)}$

▶ Interestingly, $\mathbb{D}^2$ is not a SYMMETRIC MATRIX ...

▶ Consider an ELLIPTIC BOUNDARY VALUE PROBLEM (BVP) :

$$-\nu u'' + au' + bu = f, \qquad \text{in } [-1, 1]$$
$$\alpha_- u + \beta_- u' = g_- \qquad x = -1$$
$$\alpha_+ u + \beta_+ u' = g_+ \qquad x = 1$$

▶ Chebyshev polynomials do not satisfy homogeneous boundary conditions, hence standard Galerkin approach is not directly applicable.

▶ BASIS RECOMBINATION :

  ▶ Convert the BVP to the corresponding form with HOMOGENEOUS BOUNDARY CONDITIONS

  ▶ Take linear combinations of Chebyshev polynomials to construct a new basis satisfying HOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS $\varphi_k(\pm 1) = 0$

$$\varphi_k(x) = \begin{cases} T_k(x) - T_0(x) = T_k - 1, & k - \text{even} \\ T_k(x) - T_1(x), & k - \text{odd} \end{cases}$$

  Note that the new basis preserves orthogonality

► THE TAU METHOD (Lanczos, 1938) consists in using a Galerkin approach in which explicit enforcement of the boundary conditions replaces projections on some of the test functions

► Consider the residual

$$R_N(x) = -\nu u_N'' + a u_N' + b u_N - f,$$

where $u_N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$

► Cancel projections of the residual on the first $N-2$ basis functions

$$(R_N, T_l)_\omega = \sum_{k=0}^{N} \left(-\nu \hat{u}_k^{(2)} + a \hat{u}_k^{(1)} + b \hat{u}_k \right) \int_{-1}^{1} T_k T_l \omega \, dx - \int_{-1}^{1} f T_l \omega \, dx, \quad l = 0, \ldots, N-2$$

► Thus, using orthogonality, we obtain

$$-\nu \hat{u}_k^{(2)} + a \hat{u}_k^{(1)} + b \hat{u}_k = \hat{f}_k, \quad k = 0, \ldots, N-2$$

where $\hat{f}_k = \int_{-1}^{1} f T_k \omega \, dx$

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- Noting that $T_k(\pm 1) = (\pm 1)^k$ and $T_k'(\pm 1) = (\pm 1)^{k+1} k^2$, the BOUNDARY CONDITIONS are enforced by supplementing the residual equations with

$$\sum_{k=0}^{N} (-1)^k (\alpha_- - \beta_- k^2) \hat{u}_k = g_-$$

$$\sum_{k=0}^{N} (-1)^k (\alpha_+ + \beta_+ k^2) \hat{u}_k = g_+$$

- Expressing $\hat{u}_k^{(2)}$ and $\hat{u}_k^{(1)}$ in terms of $\hat{u}_k$ via the Chebyshev spectral differentiation matrices we obtain the following system

$$\mathbb{A}\hat{U} = \hat{F}$$

where $\hat{U} = [\hat{u}_0, \ldots, \hat{u}_N]^T$, $F = [\hat{f}_0, \ldots, \hat{f}_{N-2}, g_-, g_+]$ and the matrix $\mathbb{A}$ is obtained by adding the two rows representing the boundary conditions (see above) to the matrix $\mathbb{A}_1 = -\nu \hat{\mathbb{D}}^2 + a \hat{\mathbb{D}} + bI$.

- When the domain boundary is not just a point (e.g., in 2D / 3D), formulation of the Tau method becomes somewhat more involved

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▶ Consider the residual

$$R_N(x) = -\nu u_N'' + a u_N' + b u_N - f,$$

where $u_N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$

▶ Cancel this residual at $N-1$ GAUSS-LOBATTO-CHEBYSHEV collocation points located in the interior of the domain

$$-\nu u_N''(x_j) + a u_N'(x_j) + b u_N(x_j) = f(x_j), \quad j = 1, \ldots, N-1$$

▶ Enforce the two boundary conditions at endpoints

$$\alpha_- u_N(x_N) + \beta_- u_N'(x_N) = g_-$$
$$\alpha_+ u_N(x_0) + \beta_+ u_N'(x_0) = g_-$$

Note that this shows the utility of using the
GAUSS-LOBATTO-CHEBYSHEV collocation points

▶ Consequently, the following system of $N+1$ equations is obtained

$$\sum_{k=0}^{N}(-\nu d_{jk}^{(2)} + a d_{jk}^{(1)})u_N(x_j) + b u_N(x_j) = f(x_j), \;\; j = 1, \ldots, N-1$$

$$\alpha_- u_N(x_N) + \beta_- \sum_{k=0}^{N} d_{Nk}^{(1)} u_N(x_k) = g_-$$

$$\alpha_+ u_N(x_0) + \beta_+ \sum_{k=0}^{N} d_{0k}^{(1)} u_N(x_k) = g_+$$

which can be written as $\mathbb{A}_c U = F$, where $[\mathbb{A}_c]_{jk} = [\mathbb{A}_{c0}]_{jk}$,
$j, k = 1, \ldots, N-1$ with $\mathbb{A}_{c0}$ given by

$$\mathbb{A}_{c0} = (-\nu \mathbb{D}^2 + a\mathbb{D} + b\mathbb{I})U$$

and the BOUNDARY CONDITIONS above added as the rows 0 and N of $\mathbb{A}_c$

▶ Note that the matrix corresponding to this system of equations may be POORLY CONDITIONED , so special care must be exercised when solving this system for large N.

▶ Similar approach can be used when the nodal values $u(x_j)$, rather than the Chebyshev coefficients $\hat{u}_k$ are unknowns

▶ When the equation has NONCONSTANT COEFFICIENTS , similar difficulties as in the Fourier case are encountered (evaluation of CONVOLUTION SUMS)

▶ Consequently, the COLLOCATION (pseudo-spectral) approach is preferable along the guidelines laid out in the case of the Fourier spectral methods

▶ Assuming $a = a(x)$ in the elliptic boundary value problem, we need to make the following modification to $\mathbb{A}_c$:

$$\mathbb{A}'_{c0} = (-\nu \mathbb{D}^2 + \mathbb{D}' + b\mathbb{I})U,$$

where $\mathbb{D}' = [a(x_j)d^{(1)}_{jk}], j, k = 1, \ldots, N$

▶ For the Burgers equation $\partial_t u + \frac{1}{2}\partial_x u^2 - \nu \partial_x^2 u$ we obtain at every time step n

$$(\mathbb{I} - \Delta t \, \nu \, \mathbb{D}^{(2)})U^{n+1} = U^n - \frac{1}{2}\Delta t \, \mathbb{D} \, W^n,$$

where $[W^n]_j = [U^n]_j[U^n]_j$; Note that an algebraic system has to be solved at each time step

Epilogue — Domain Decomposition

- Motivation:
  - treatment of problem in IRREGULAR DOMAINS
  - STIFF PROBLEMS
- PHILOSOPHY — partition the original domain Ω into a number of SUBDOMAINS $\{\Omega_m\}_{m=1}^{M}$ and solve the problem separately on each those while respecting consistency conditions on the interfaces
- Spectral Element Method
  - consider a collection of problems posed on each subdomain Ω_m
    $$\mathcal{L}u_m = f$$
    $$u_{m-1}(a_m) = u_m(a_m), \qquad u_m(a_{m+1}) = u_{m+1}(a_{m+1})$$
  - Transform each subdomain Ω_m to $I = [-1, 1]$
  - use a separate set of N_m ORTHOGONAL POLYNOMIALS to approximate the solution on every subinterval
  - boundary conditions on interfaces provide coupling between problems on subdomains