HOMEWORK #2

Due: November 17 (Thursday) by midnight

Submit your solutions in the form of a *single* MATLAB m-file via Email to the instructor. Please embed your narrative answers in this file using the instruction `disp(...)`, or submit them as a separate document in the PDF format (no Word files will be accepted!). Late submissions will not be considered.

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1. You are given the following two functions $f_1, f_2 \; : \; [0, 2\pi] \rightarrow \mathbb{R}$

$$f_1(x) = e^{\sin(x)},$$
$$f_2(x) = e^{\sin\left(\frac{1}{2}x\right)}.$$

Assuming that these functions are periodic on $\mathbb{R}$ (with period 2π), approximate numerically the definite integrals $\int_0^{2\pi} f_1(x)\,dx$ and $\int_0^{2\pi} f_2(x)\,dx$ using a suitable *periodic* Gaussian quadrature. Plot the relative error (with respect to the exact values of the integrals) as a function of the number N of subintervals discretizing $[0, 2\pi]$, where $2^3 \le N \le 2^{16}$. What is the order of accuracy of the quadrature in the two cases? What explains the difference?

The exact value of $\int_0^{2\pi} f_1(x)\,dx$ can be found analytically (for example, using Maple), whereas $\int_0^{2\pi} f_2(x)\,dx = 12.4175160714222$.
[4 points]

2. Consider the periodic extensions (with period 2π) of the following two functions

$$g_1(x) = \begin{cases} -1 & 0 \le x \le \pi \\ 1 & \pi < x \le 2\pi \end{cases},$$
$$g_2(x) = \begin{cases} \dfrac{2}{\pi}x & 0 \le x \le \dfrac{\pi}{2} \\ -\dfrac{2}{\pi}x + 2 & \dfrac{\pi}{2} < x \le \dfrac{3\pi}{2} \\ \dfrac{2}{\pi}x - 4 & \dfrac{3\pi}{2} < x \le 2\pi \end{cases}.$$

Construct truncated Fourier series in the form

$$g_i^N(x) = \sum_{k=1}^{N} \frac{\hat{g}_{i,k}}{\sqrt{\pi}} \sin(kx), \qquad i = 1, 2, \tag{1}$$

where the Fourier coefficients $\hat{g}_{i,k} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} g_i(x) \sin(kx)\,dx$, $k = 1, \dots, N$, can be approximated with the periodic Gaussian quadrature using $N_0 = 10^5$ grid points. Then plot

(a) the magnitudes $|\hat{g}_{i,k}|$, $i = 1, 2$, of the Fourier coefficients as a function of k assuming that the number of terms in series (1) is $N = 100$ (one figure),

(b) the relative error norms

$$E_2 = \frac{\|g_i^N - g_i\|_{L_2(0,2\pi)}}{\|g_i\|_{L_2(0,2\pi)}}, \qquad E_\infty = \frac{\|g_i^N - g_i\|_{L_\infty(0,2\pi)}}{\|g_i\|_{L_\infty(0,2\pi)}}$$

as functions of the number of terms $N = 1, \dots, 100$ in Fourier series (1) for $i = 1, 2$; the norms $\| \cdot \|_{L_2(0,2\pi)}$ and $\| \cdot \|_{L_\infty(0,2\pi)}$ can be approximated with, respectively, the MATLAB functions `norm(vec,2)` and `norm(vec,inf)`, where `vec` represents a function discretized on an uniform grid with N_0 points (separate figure for each norm),

(c) the functions $g_i(x)$ and the corresponding truncated Fourier series $g_i^N(x)$ for $N = 1, 21, 41, \dots$ and $i = 1, 2$ as functions of $x \in [0, 2\pi]$ (separate figure for each function).

Explain the behaviour of the errors E_2 and E_∞ with respect to the number of terms N in series (1) for the two functions g_1 and g_2.
[6 points]

3. [**Bonus Question**] Construct and plot a numerical approximation of a periodic function with period 2π and with the *minimum* regularity necessary to prevent Runge's phenomenon. Note that such function should possess square-integrable derivatives of (fractional) order $1/2 + \epsilon$, where $\epsilon > 0$.
[2 points]