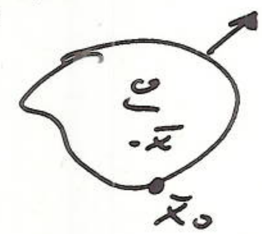
BOUNDARY ELEMENT METHOD (BEM)

We will consider an alternative approach to the solution of elliptic (and other) PDE problems.

Suppose we want to solve Dirichlet problem for the Laplace eq. in 2D: $\underline{\Omega} \subset \mathbb{R}^2$



$$(*) \quad \begin{cases} \Delta u = 0 & \text{in } \underline{\Omega} \\ u|_{\partial\underline{\Omega}} = g \end{cases}$$

Let's start with the following Green's identity (integration by parts)

$$\int_{\underline{\Omega}} (v\Delta w - w\Delta v)\, d\underline{\Omega} = \oint_{\partial\underline{\Omega}} \left(v\frac{\partial w}{\partial n} - w\frac{\partial v}{\partial n}\right) d\sigma$$

and assume:

- v is a harmonic function on $\underline{\Omega}$: $\Delta v = 0$
- $\Delta w = \delta(\bar{x} - \bar{x}')$ ($\Rightarrow$ w is the fundamental sol'n)

$$v(\bar{x}) = \int_{\underline{\Omega}} v\,\delta(x - x')\, d\underline{\Omega}(x') = \oint_{\partial\underline{\Omega}} \left(v\frac{\partial w}{\partial n} - w\frac{\partial v}{\partial n}\right) d\sigma$$

Thus, any <u>harmonic</u> function can be expressed as

$$v(\bar{x}) = \oint_{\partial\underline{\Omega}} \left[v\,\frac{\partial G(\bar{x},\bar{x}')}{\partial n} - G(\bar{x},\bar{x}')\frac{\partial v}{\partial n} \right] d\sigma(\bar{x}')$$

Or, alternatively

$$v(\bar{x}) = \oint_{\partial \underline{\Omega}} \mu(\bar{x}') \frac{\partial G(\bar{x},\bar{x}')}{\partial n} d\sigma(\bar{x}')$$

$$= \oint_{\partial \underline{\Omega}} \eta(\bar{x}') \, G(\bar{x},\bar{x}') \, d\sigma(\bar{x}')$$

η, μ - single/double layer potential

To fix attention, let's adopt the second representation

Thus $\quad u = \oint_{\partial \underline{\Omega}} \eta \, G \, d\sigma \quad , \; G(\bar{x},\bar{x}') = -\frac{1}{2\pi} \ln|\bar{x}-\bar{x}'|$

by construction satisfies the equation $\Delta u = 0$, but involves an undetermined function η defined on the domain boundary.

What about the boundary condition $u|_{\partial \underline{\Omega}} = g$?

Take the limit $\bar{x} \to \bar{x}_0 \in \partial \underline{\Omega}$

The logarithmic kernel becomes singular, but the entire integrand remains integrable and yields

(**)

$$\frac{1}{2\pi} \oint_{\partial \underline{\Omega}} \eta(\bar{x}') \ln|\bar{x}_0 - \bar{x}'| \, d\sigma(\bar{x}') = g(\bar{x}_0) \quad , \; \bar{x}_0 \in \partial \underline{\Omega}$$

from which one can find the distribution of the single-layer potential η on the boundary.

Then, once the potential is known, one can use

$$u(\bar{x}) = \frac{1}{2\pi} \oint_{\partial\Omega} \eta(\bar{x}') \ln|\bar{x}-\bar{x}'| \, dS(\bar{x}')$$

to evaluate the solution anywhere in the domain Ω.

Remarks

* Instead of solving problem ⊛ on a 2D domain (with a sparse matrix), one has to solve integral equation on a 1D domain boundary (with a dense matrix)
* no need to mesh the interior of the domain Ω.
* depending on the domain (interior vs. exterior) and the BCs, there are many different versions of BEM.

PRESSURE CORRECTION METHOD

How to ~~[illegible]~~ enforce incompressibility in the integration of the Navier-Stokes equation

(time-)

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u} = -\nabla p + \frac{1}{Re}\Delta\bar{u}$$

$$\nabla\cdot\bar{u} = 0$$

Pressure correction / splitting method → workhorse for commercial codes

For simplicity only, assume hybrid explicit/implicit treatment of different terms

$$\frac{\tilde{u} - \bar{u}^n}{\Delta t} = -(\bar{u}^n \cdot \nabla)\bar{u}^n - \nabla p^n + \frac{1}{Re}\Delta \tilde{u} \Rightarrow \tilde{u}$$

The intermediate field $\tilde{u}$ need not be div-free

Introduce pseudo-pressure φ

$$\Delta \varphi = \nabla \cdot \tilde{u}$$
$$+ \text{BCs}$$

Then, the corrected velocity field $\bar{u}^{n+1}$ at the new time level is

$$\bar{u}^{n+1} = \tilde{u} - \nabla \varphi$$

Note that $\nabla \cdot \bar{u}^{n+1} = \nabla \cdot \tilde{u} - \underbrace{\nabla \cdot \nabla}_{\Delta} \varphi = 0$

New pressure $p^{n+1} = \varphi$

<u>Remark</u>

The problem of advancing Navier-Stokes system in time breaks down into a sequence of elliptic solves (details depend on specific space discretization)