HOMEWORK #1

Due: October 9 (Wednesday) by midnight

Submit your solutions, i.e., your brief report in the form of a *single* PDF file (no Word files will be accepted!) and, if applicable, your MATLAB code(s) in the form of a *single* m-file via Email to the instructor. Late submissions will not be considered.

The total number of points is 20.

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1. Assuming that $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are differentiable vector fields, prove the following identities

   (a) $$\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}$$

   (b) $$\frac{1}{2}(\nabla \times \mathbf{a}) \times \mathbf{b} = \mathbf{R}\mathbf{b},$$

   where $[\mathbf{R}]_{ij} = \frac{1}{2}\left[\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i}\right]$, $i, j = 1, 2, 3$. Express your proofs using the index notation with Einstein's summation convention.
   [3+2 = 5 points]

2. Assuming that the fluid flow is

   - time-independent and two-dimensional,
   - characterized by constant density $\rho = \text{Const}$,
   - subject to potential body forces only, and
   - described by the velocity vector field $\mathbf{u} = \mathbf{u}(\mathbf{x})$

   derive an equation describing the variation of the Bernoulli constant H in space.
   [3 points]

3. What form do the streamlines characterizing the following two velocity fields have

   (a)
   $$\mathbf{u}(x, y) = \left[\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right], \qquad x^2 + y^2 > 0$$

   (b)
   $$\mathbf{u}(x, y) = [-y, x]?$$

   How do these fields differ in terms of the Helmholz-Hodge decomposition?
   [4 points]

4. Consider a two-dimensional irrotational and incompressible flow of an ideal fluid in the domain Ω shown in Figure 1. The velocity field $\mathbf{u} : \Omega \to \mathbb{R}^2$ satisfies the following boundary conditions

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= 0 && \text{on } \Gamma_0, \\ \mathbf{u} \cdot \mathbf{n} &= -V && \text{on } \Gamma_1, \\ \mathbf{u} \cdot \mathbf{n} &= V && \text{on } \Gamma_2, \end{aligned}$$

where $\mathbf{n}$ is the unit normal vector oriented as shown in Figure 1 and $V > 0$ is a given number.

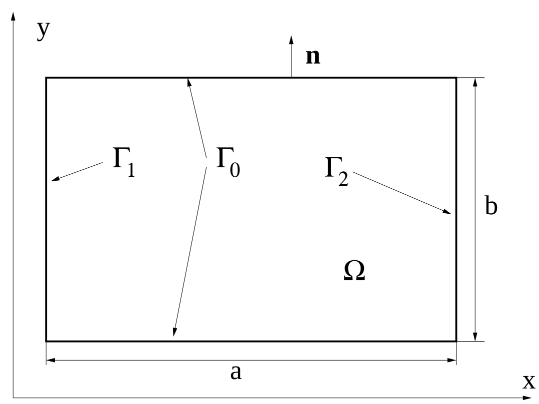

Figure 1: Flow set-up in Problem 4.

Assuming that the velocity fields can be represented as

(a)
$$\mathbf{u} = \nabla\phi,$$

(b)
$$\mathbf{u} = \nabla^{\perp}\psi,$$

formulate the partial differential equations satisfied by ϕ and ψ together with suitable boundary conditions. Solve these equations to obtain ϕ and ψ, and then use these fields to compute the velocity field $\mathbf{u}$.
[8 points]