MATH 749

HOMEWORK ASSIGNMENT #1 – SOLUTIONS

1) Use the index notation and Einstein summation convention

a) start with the terms with the curl

$$\left[\bar{a} \times (\nabla \times \bar{b})\right]_i = \varepsilon_{ijk}\, a_j\, \varepsilon_{kpq} \frac{\partial b_q}{\partial x_p}$$

$$= \varepsilon_{ijk}\varepsilon_{kpq}\, a_j \frac{\partial b_q}{\partial x_p} \underset{\uparrow}{=} \varepsilon_{kij}\varepsilon_{kpq}\, a_j \frac{\partial b_q}{\partial x_p} =$$

$$\varepsilon_{ijk} = \varepsilon_{kij}$$

Now use the following property of the Levi-Civita tensor

$$\varepsilon_{kij}\varepsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\, a_j \frac{\partial b_q}{\partial x_p} = a_q \frac{\partial b_q}{\partial x_i} - a_p \frac{\partial b_i}{\partial x_p}$$

change $q \to p$ in the given term

$$= a_p \frac{\partial b_p}{\partial x_i} - a_p \frac{\partial b_i}{\partial x_p}$$

Likewise:

$$\left[\bar{b} \times (\nabla \times \bar{a})\right]_i = b_p \frac{\partial a_p}{\partial x_i} - b_p \frac{\partial a_i}{\partial x_p}$$

The other two terms on the RHS

$$\left[(\bar{a}\cdot\nabla)\bar{b}\right]_i = a_p\frac{\partial b_i}{\partial x_p} \;,\; \left[(\bar{b}\cdot\nabla)\bar{a}\right]_i = b_p\frac{\partial a_i}{\partial x_p}$$

Putting together all the terms on the RHS

$$\left[\bar{a}\times(\nabla\times\bar{b}) + \bar{b}\times(\nabla\times\bar{a}) + (\bar{a}\cdot\nabla)\bar{b} + (\bar{b}\cdot\nabla)\bar{a}\right]_i =$$

$$= a_p\frac{\partial b_p}{\partial x_i} - \cancel{a_p\frac{\partial b_i}{\partial x_p}} + b_p\frac{\partial a_p}{\partial x_i} - \cancel{b_p\frac{\partial a_i}{\partial x_p}}$$

$$+ \cancel{a_p\frac{\partial b_i}{\partial x_p}} + \cancel{b_p\frac{\partial a_i}{\partial x_p}} = \frac{\partial}{\partial x_i}(a_p b_p)$$

$$= \left[\nabla(\bar{a}\cdot\bar{b})\right]_i$$

■

b) $$\left[\frac{1}{2}(\nabla\times\bar{a})\times\bar{b}\right]_i = \frac{1}{2}\,\varepsilon_{pqm}\frac{\partial a_m}{\partial x_q}\,\varepsilon_{ipk}\,b_k$$

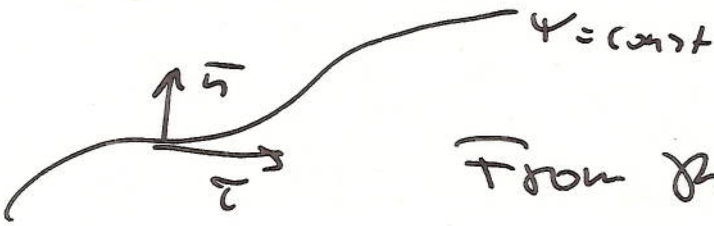
$$= -\frac{1}{2}\,\varepsilon_{pqm}\,\varepsilon_{pik}\frac{\partial a_m}{\partial x_q}b_k = -\frac{1}{2}\left(\delta_{qi}\delta_{mk} - \delta_{qk}\delta_{mi}\right)\frac{\partial a_m}{\partial x_q}b_k$$

$$= -\frac{1}{2}\left(\frac{\partial a_k}{\partial x_i} - \frac{\partial a_i}{\partial x_k}\right)b_k = \underbrace{\frac{1}{2}\left(\frac{\partial a_i}{\partial x_k} - \frac{\partial a_k}{\partial x_i}\right)}_{R_{ik}}b_k$$

$$= \left[\bar{\bar{R}}\,\bar{b}\right]_i$$

■

2) WLOG consider 2D case and introduce a local coordinate system attached to the streamlines



From the momentum equation

$$\nabla\underbrace{\left(\frac{1}{\rho}p + \frac{1}{2}|\bar{u}|^2 + \Phi\right)}_{H = H(n)} = \bar{u}\times(\nabla\times\bar{u})$$

$$\nabla = \bar{\tau}\frac{\partial}{\partial s} + \bar{n}\frac{\partial}{\partial n}$$

$$\bar{u}\times(\nabla\times\bar{u}) = \left[\bar{n}\cdot(\bar{u}\times(\nabla\times\bar{u}))\right]\bar{n} + 0\,\bar{\tau}$$

$$\nabla H = \bar{\tau}\underbrace{\frac{\partial}{\partial s}H}_{0} + \bar{n}\frac{\partial}{\partial n}H = \left[\bar{n}\cdot(\bar{u}\times(\nabla\times\bar{u}))\right]\bar{n}$$

Thus:

$$\underbrace{\frac{\partial}{\partial n}}_{\text{derivative in the direction normal to the stream-function}} H = \bar{n}\cdot(\bar{u}\times(\nabla\times\bar{u}))$$

4) [Solution to problem 3 on the last page]

a) $\bar{u} = \nabla \phi$ in $\underline{\Omega}$

Equation satisfied by the potential:

$$\left.\begin{aligned} &\nabla \cdot \bar{u} = 0 \quad \Rightarrow \quad \Delta \phi = 0 \quad \text{in } \underline{\Omega} \\ &\text{BCs:} \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{at } y = 0, b \\ &\qquad \frac{\partial \phi}{\partial n} = -\frac{\partial \phi}{\partial x} = -V \quad \text{at } x = 0 \\ &\qquad \frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} = V \quad \text{at } x = a \end{aligned}\right\} \ (*)$$

The solution of Neumann problem (*) is ~~...~~

$$\phi(x,y) = Vx + \phi_0, \quad \phi_0 - \text{arbitrary constant}$$

The velocity field

$$\bar{u} = \nabla \phi = \underline{[V, 0]}$$

b) $\bar{u} = \nabla^{\perp} \psi$ in $\underline{\Omega}$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \Rightarrow \nabla^{\perp} \nabla^{\perp} \psi = \Delta \psi = 0$$

BCs

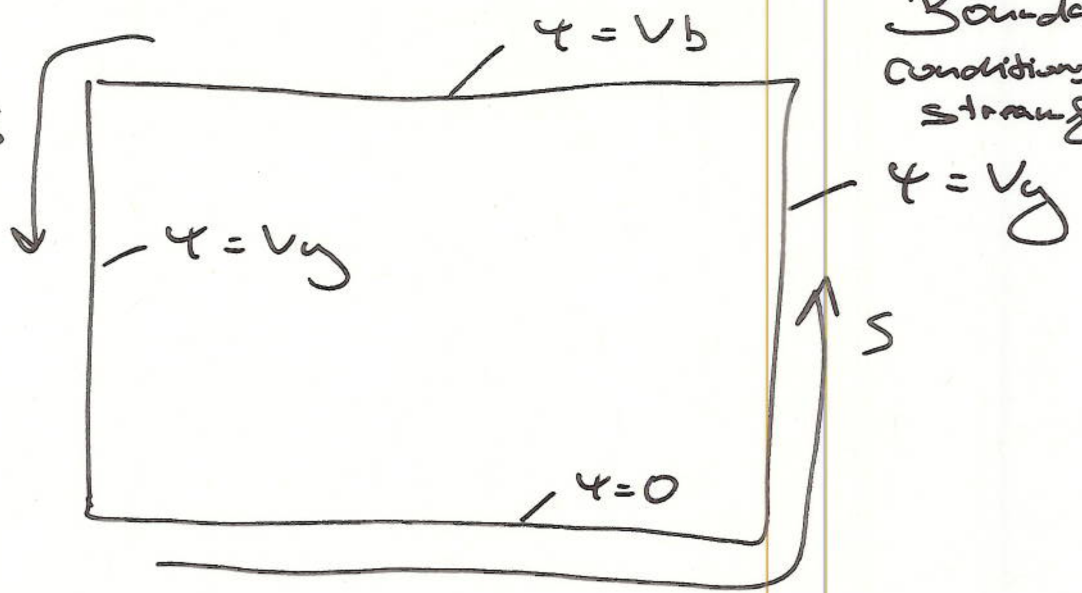
at ~~...~~

$y = 0 \quad \nabla^{\perp}\psi \cdot \bar{n} = \dfrac{\partial \psi}{\partial s} = 0 \Rightarrow \psi = \psi_0 = 0$ (arbitrary constant)

$x = a \quad \nabla^{\perp}\psi \cdot \bar{n} = \dfrac{\partial \psi}{\partial s} = V$

$$\psi(y) = \int_0^y V \, ds' = Vy + \psi_0 = Vy$$

Boundary conditions for streamfunction ψ



$$y = b \quad \nabla^{\perp}\psi \cdot \bar{n} = \frac{\partial \psi}{\partial s} = 0 \Rightarrow \psi = \text{const} = Vb$$

ensures continuity

$$x = 0 \quad \nabla^{\perp}\psi \cdot \bar{n} = \frac{\partial \psi}{\partial s} = -V$$

$$\psi(y) = -\int_0^{b-y} V\, ds' = \left| \begin{array}{l} s' = b - y' \\ ds' = -dy' \end{array} \right|$$

$$= V\int_b^y dy' = V(y-b) + \underbrace{Vb}_{\psi_0} = Vy$$

Thus, ψ satisfies the following Dirichlet problem

$$\begin{cases} \Delta \psi = 0 & \text{in } \underline{\Omega} \\ \psi = Vy & \text{at } x = 0, a \\ \psi = 0 & \text{at } y = 0 \\ \psi = Vb & \text{at } y = b \end{cases}$$

The solution is $\psi(x,y) = Vy$

so that $\bar{u} = \nabla^{\perp}\psi = \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right] = [V, 0]$

3) ~~Streamlines~~ Equation for the streamlines

$$\frac{dx}{u} = \frac{dy}{v}$$

a) $$\frac{dx}{-y/\cancel{x^2+y^2}} = \frac{dy}{x/\cancel{x^2+y^2}} \Rightarrow x\,dx = -y\,dy$$

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 = C \Rightarrow x^2+y^2 = C'$$

The streamlines are circles

b) $$\frac{dx}{-y} = \frac{dy}{x} \Rightarrow x\,dx = -y\,dy \Rightarrow x^2+y^2 = C'$$

The streamlines are <u>also circles</u>

Field (a) is irrotational $\left(\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = 0$$

Field (b) has nonvanishing vorticity

$$\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 2 \neq 0$$