HOMEWORK #2

Due: November 6 (Wednesday) by midnight

Submit your solutions, i.e., your brief report in the form of a *single* PDF file (no Word files will be accepted!) and, if applicable, your `Matlab` or `Maple` code(s) in the form of a *single* m/mw-file via Email to the instructor. Late submissions will not be considered.

The total number of points is 20.

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1. Consider a 2D vortex sheet which at the given instant of time coincides with the segment $[-1, 1]$ of the plane. The circulation is distributed uniformly along the sheet with density $\gamma > 0$, such that

$$\int_{-1}^{1} \gamma \, ds = \gamma \cdot 2 = \Gamma,$$

where Γ is the total circulation of the sheet.

   (a) Obtain expressions for the velocity field $[u(x, y), v(x, y)]$ induced by this sheet.

   (b) How does its magnitude $|\mathbf{u}| = \sqrt{u^2 + v^2}$ behave when the point (x, y) where it is evaluated

      i. approaches the center $(0, 0)$ of the sheet,

      ii. approaches the endpoint $(1, 0)$ of the sheet?

   Hint: it is advisable to use a symbolic algebra package such as `Maple` or the symbolic toolbox in `Matlab`.
   [7 points]

2. Consider two point vortices distance $L > 0$ apart with

   (a) opposite circulations $\Gamma_1 = \Gamma$ and $\Gamma_2 = -\Gamma$, and

   (b) identical circulations $\Gamma_1 = \Gamma_2 = \Gamma$,

   where $\Gamma > 0$ is given. Knowing that these vortices represent a *relative* equilibrium, i.e., a fixed point in a suitable moving frame of reference, of system

$$\frac{d\overline{z}_j}{dt} = -\frac{i}{2\pi} \sum_{k=1,\ k\neq j}^{N} \frac{\Gamma_k}{z_j - z_k}, \quad j = 1, \dots, N,$$

   where $N = 2$ and $z_1(t), z_2(t) \in \mathbb{C}$ are the positions of the vortices, derive a relation between their (a) translation and (b) rotation velocity and the parameters Γ and L. What is the velocity in the absolute frame of reference of the point located half-way between the point vortices in the two cases?
   [7 points]

3. Consider the potential flow past a circular cylinder with unit radius and corresponding to the uniform flow at infinity $U = 1$. The complex potential characterizing such a flow is
$$W(z) = \left(z + \frac{1}{z}\right) - \frac{i\Gamma}{2\pi}\ln z,$$
where Γ is the circulation. Determine the range of values of Γ for which the flow admits precisely *two* stagnation points (i.e., points on the boundary $|z| = 1$ where the velocity vanishes).
[6 points]