MATH 749 – HOMEWORK ASSIGNMENT #2

SOLUTIONS

PROBLEM 1

```
> restart;
> with(plots):
> assume(g>0);
```

Fundamental Solution

```
> G := (1/(2*Pi))*ln(sqrt( (x-x0)^2 + (y-y0)^2 ));
```

$$G := \frac{1}{4}\,\frac{\ln\left((x-x0)^2+(y-y0)^2\right)}{\pi} \qquad (1)$$

Streamfunction

```
> G0 := subs(y0=0,G);
```

$$G0 := \frac{1}{4}\,\frac{\ln\left((x-x0)^2+y^2\right)}{\pi} \qquad (2)$$

```
> psi := Int(-g*G0,x0=-1..1);
```

$$\psi := \int_{-1}^{1}\left(-\frac{1}{4}\,\frac{g\sim\ln\left((x-x0)^2+y^2\right)}{\pi}\right)\mathrm{d}x0 \qquad (3)$$

```
> psi := simplify(value(psi), size);
```

$$\psi := \frac{1}{4}\,\frac{1}{\pi}\Big(g\sim\Big((x-1)\ln\left(x^2+y^2-2x+1\right)+(-x-1)\ln\left(x^2+y^2+2x+1\right)$$
$$+2y\arctan\left(\frac{x-1}{y}\right)-2y\arctan\left(\frac{x+1}{y}\right)+4\Big)\Big) \qquad (4)$$

Velocity components

```
> u := simplify(diff(psi,y),size);
```

$$u := \frac{1}{2}\,\frac{g\sim\left(\arctan\left(\frac{x-1}{y}\right)-\arctan\left(\frac{x+1}{y}\right)\right)}{\pi} \qquad (5)$$

```
> v := simplify(  - diff(psi,x), size);
```

$$v := -\frac{1}{4}\,\frac{g\sim\left(\ln\left(x^2+y^2-2x+1\right)-\ln\left(x^2+y^2+2x+1\right)\right)}{\pi} \qquad (6)$$

WLOG, setting unit circulation density

```
> psi1 := subs(g=1,psi); u1 := subs(g=1,u); v1  := subs(g=1,v);
```

$$\psi 1 := \frac{1}{4}\,\frac{1}{\pi}\Big((x-1)\ln\left(x^2+y^2-2x+1\right)+(-x-1)\ln\left(x^2+y^2+2x+1\right)$$
$$+2y\arctan\left(\frac{x-1}{y}\right)-2y\arctan\left(\frac{x+1}{y}\right)+4\Big)$$

$$u1 := \frac{1}{2}\,\frac{\arctan\left(\frac{x-1}{y}\right)-\arctan\left(\frac{x+1}{y}\right)}{\pi}$$

(7)

$$v1 := -\frac{1}{4}\,\frac{\ln(x^2+y^2-2x+1)-\ln(x^2+y^2+2x+1)}{\pi} \tag{7}$$

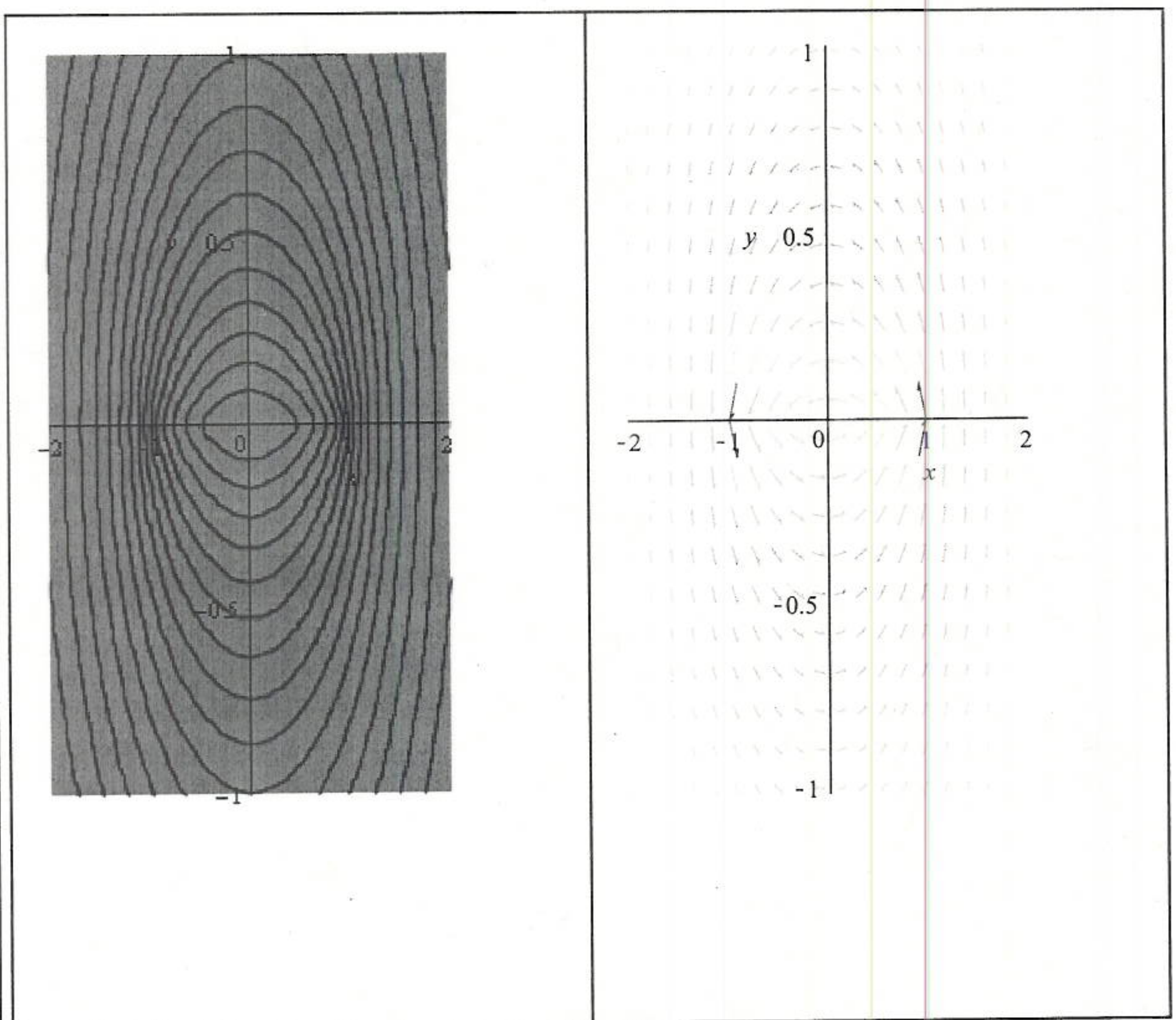
```
> P1 := contourplot(psi1,x=-2..2,y=-1..1,contours=15,filledregion=
  true);
```

$$P1 := PLOT(\ldots) \tag{8}$$

```
> P2 := fieldplot([u1,v1],x=-2..2,y=-1..1,color=sqrt(u1^2+v1^2));
```

$$P2 := PLOT(\ldots) \tag{9}$$

```
> display(array([P1,P2]));
```



Velocity magnitude

```
> V := simplify(sqrt(u1^2+v1^2),size);
```

$$V := \frac{1}{4} \tag{10}$$

$$\left(\frac{1}{\pi^2}\left(\ln(x^2+y^2-2x+1)^2-2\ln(x^2+y^2-2x+1)\ln(x^2+y^2+2x+1)\right.\right.$$

$$\left.\left.+\ln(x^2+y^2+2x+1)^2+4\left(\arctan\left(\frac{x-1}{y}\right)-\arctan\left(\frac{x+1}{y}\right)\right)^2\right)\right)^{1/2}$$

Limiting behavior

as (x,y) approches (0,0)

```
> V0 := limit(subs(x=0,V),y=0);
```

$$V0 := \frac{1}{2} \qquad (11)$$

as (x,y) approches (0,1)

- along the line x=1

```
> V1a := limit(subs(x=1,V),y=0);
```

$$V1a := \infty \qquad (12)$$

- along the line y=0

```
> V1b := limit(limit(V,y=0),x=1) assuming x>1;
```

$$V1b := \infty \qquad (13)$$

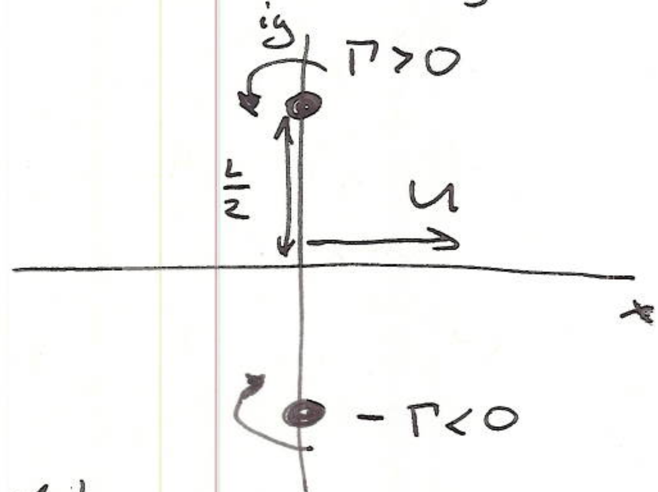
PROBLEM 2

(a) opposite circulations

$\Downarrow$

counterrotating / translating pair

WLOG, introduce coordinate system



U - translation velocity

Need to relate U to Γ and L

Transform to moving (translating) frame of reference in which equilibrium attains.

~~Consider equation for~~

$$z_1(t) = \zeta_1 + Ut \quad , \quad z_2(t) = \zeta_2 + Ut$$

Consider now the first vortex (the equations for the second will be identical)

$$\frac{dz_1}{dt} = \underbrace{\frac{d\zeta_1}{dt}}_{0} + U$$

$$\frac{d\bar{z}_1}{dt} = U = -\frac{i}{2\pi}\frac{\Gamma_2}{z_1 - z_2} =$$

$$\left|\begin{array}{l} \Gamma_2 = -\Gamma \\ z_1 = i\frac{L}{2} + Ut \\ z_2 = -i\frac{L}{2} + Ut \end{array}\right|$$

$$= \frac{i}{2\pi}\frac{\Gamma}{iL} = \frac{\Gamma}{2\pi L}$$

Thus: $\underline{\underline{U = \frac{\Gamma}{2\pi L}}}$

The midpoint $z_0 = \dfrac{z_1(t) + z_2(t)}{2} = Ut$

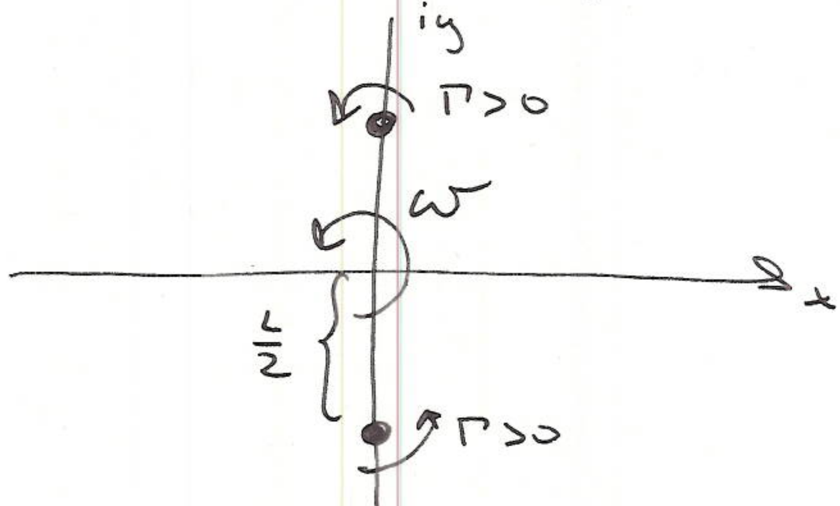
moves with the velocity $\bar{u} = [U, 0]$.

(b) same-sign circulations

$\Downarrow$

corotating pair

coordinate system



ω - rotation velocity
(rate of rotation)

Transforming to the rotating (with rate ω) frame of reference

$$z_1(t) = \zeta_1 e^{i\omega t}, \quad z_2(t) = \zeta_2 e^{i\omega t}$$

Then, proceeding as in part (a)

$$\frac{dz_1}{dt} = \zeta_1 i\omega e^{i\omega t}$$

and

$$\frac{d\bar{z}_1}{dt} = \bar{\zeta}_1 (\cancel{-}i\omega) \cancel{e^{-i\omega t}} = -\frac{i}{2\pi} \frac{\Gamma}{\zeta_1 e^{i\omega t} - \zeta_2 e^{i\omega t}}$$

$$= \cancel{-}\frac{i}{2\pi} \frac{\Gamma}{\zeta_1 - \zeta_2} \cancel{e^{-i\omega t}}$$

$$\bar{z}_1 \omega = \frac{1}{2\pi}\frac{\Gamma}{z_1 - z_2}, \quad \text{where } z_1 = i\frac{L}{2},\ z_2 = -i\frac{L}{2}$$

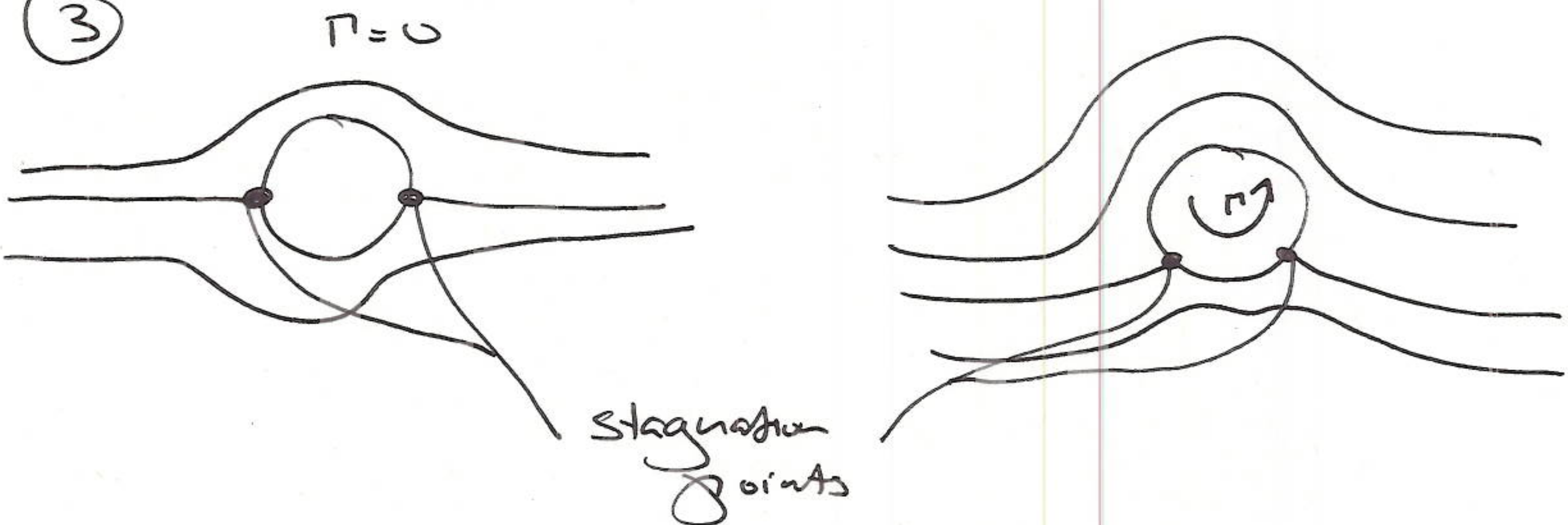
$$-\cancel{i}\frac{L}{2}\omega = \frac{1}{2\pi}\frac{\Gamma}{iL} = \frac{\Gamma}{2\pi}\frac{1}{L}\cancel{(-i)}$$

$$\omega = \frac{\Gamma}{\pi L^2}$$

The midpoint $z_0 = \dfrac{z_1(t) + z_2(t)}{2} = 0$

does <u>not move</u>

(3)



How do characterize the stagnation points?
The vanishing of the tangential velocity component
(the normal component vanishes identically)

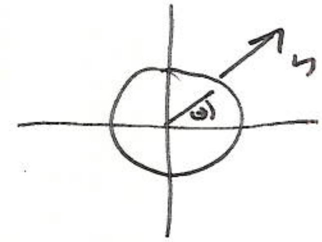
$$W(z) = \left(z + \frac{1}{z}\right) - \frac{i\Gamma}{2\pi}\ln z$$

Complex velocity

$$V(z) = \frac{dW}{dz} = 1 - \frac{1}{z^2} - \frac{i\Gamma}{2\pi}\frac{1}{z}$$

Evaluate the velocity on the boundary ($z = e^{i\theta}$)

$$V(\theta) = 1 - \frac{1}{e^{2i\theta}} - \frac{i\Gamma}{2\pi}\frac{1}{e^{i\theta}}$$



Consider normal vector $n = e^{i\theta}$

then

$$V \cdot n = (u - iv)(n_x + i n_y) = V_n - i V_\tau$$

normal tangential components

$$V\cdot n = e^{i\theta}\left(1 - e^{-2i\theta} - \frac{i\Gamma}{2\pi} e^{-i\theta}\right)$$

$$= e^{i\theta} - e^{-i\theta} - \frac{i\Gamma}{2\pi} = 2i\sin\theta - \frac{i\Gamma}{2\pi}$$

$$= i\left(2\sin\theta - \frac{\Gamma}{2\pi}\right) = 0$$

Condition on the normal velocity component satisfied identically

$$2\sin\theta = \frac{\Gamma}{2\pi} \Rightarrow$$ two solutions exist when

$$\left|\frac{\Gamma}{4\pi}\right| < 1 \Rightarrow \underline{-4\pi < \Gamma < 4\pi}$$

When $\Gamma > 4\pi$ there is no stagnation point on the boundary of the obstacle