MATH 749 – MATHEMATICAL AND COMPUTATIONAL FLUID DYNAMICS

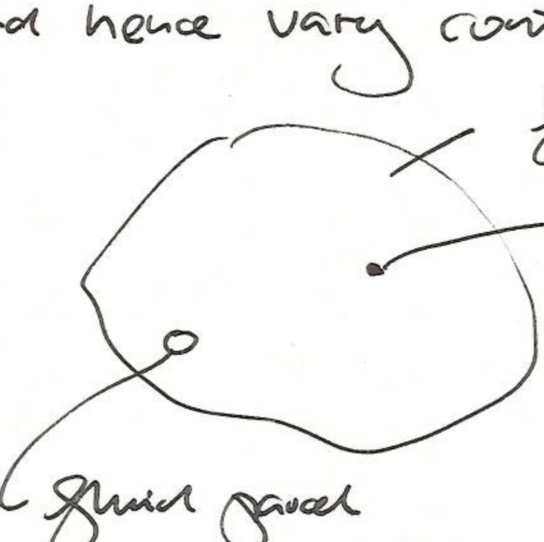
1) Conservation of Mass & Momentum

(a) Eulerian & Lagrangian Descriptions

Classical Fluid Mechanics (or Dynamics) – idealization based on the concept of continuum

In reality, fluids (liquids + gases) are made of distinct molecules, however, in continuous theories observable properties (such as velocity, pressure, density, etc.) are understood as averages over small volumes containing many molecules, and hence vary continuously(?) in space and in time

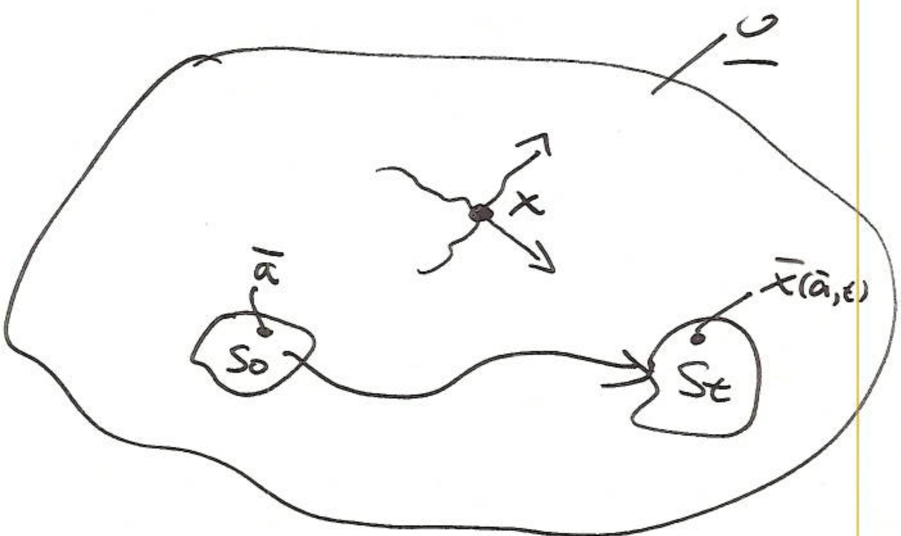
fluid region $\Omega \subseteq \mathbb{R}^N$, $N = 1, 2, 3$



fluid particle – point in the continuum → not to be confused with ~~[illegible]~~ a molecule, e.g., particle velocity $\neq$ molecule velocity

fluid parcel – small volume

Continuous theory has limitation (which are more or less well understood)



$S_0, S_t \in \underline{\mathcal{O}}$ – fluid subregions at times 0 and t

Flow map M_t: $M_t S_0 = S_t$

If $\bar{a} = (a_1, \dots, a_n) \in S_0$, then $\bar{x} = \bar{X}(\bar{a}, t)$ is the position (at time t) of the fluid particle initially located at $\bar{a}$

$\bar{X}(\bar{a}, t)$ is the Lagrangian coordinate of the fluid particle.

The independent variable $\bar{a}$ is the initial position (although it could also be any other parametrization or labelling)

* Lagrangian description studies the flow by following individual particles.

* Eulerian description focuses on the flow properties at an ~~[illegible]~~ selected point $\bar{x}$ (through which different fluid particles can pass)

* Relation between Eulerian and Lagrangian descriptions

Let $\bar{u} : \Omega \times [0,T] \to \mathbb{R}^N$ be the Eulerian velocity field; $[0,T] \ni t$ is the time interval of interest $(T > 0)$

Denote $\bar{x} = \bar{X}(\bar{a}, t)$

Then
$$\begin{cases} \dot{x}(t) = \left.\frac{\partial \bar{X}}{\partial t}\right|_a = \bar{u}(\bar{X}(a,t), t) \\ \bar{x}(0) = \bar{a} \end{cases} \qquad (1)$$

That is, the rate of change of the Lagrangian ~~[illegible]~~ coordinate is given by the Eulerian velocity field

Remark

Lagrangian (particle) trajectories can be found by solving ODE initial value problem (1), given the velocity field $\bar{u}$

Remark

When the velocity field is steady, i.e., does not depend on time $\bar{u} = \bar{u}(x)$, ~~the~~ system (1) is autonomous.

Both Lagrangian and Eulerian ~~descriptions are~~ formalisms are useful, although the description in terms of the latter tends to be mathematically more tractable.

→ examples in the textbook

* **dynamic problem** – understand how the velocity, pressure and other fields evolve in time

* **kinematic problem** – given the velocity field $\bar{u}(\bar{x},t)$, understanding its properties and how they affect particle motion

* **particle paths** or **streaklines** – Lagrangian trajectories $\bar{X}(\bar{a},t)$ obtained as solutions of problem (1)

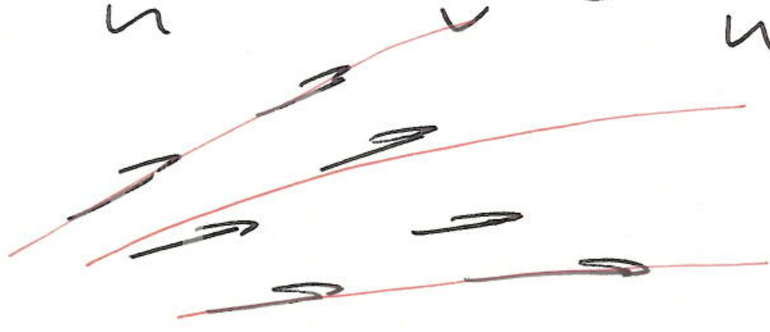
* **Instantaneous streamlines** – given a time-dependent velocity field $\bar{x}=[x,y,z]$

$$\bar{u}(\bar{x},t)=\left[u(x,y,z,t),\ v(x,y,z,t),\ w(x,y,z,t)\right]$$

they are the integral curves satisfying the equation

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

[particle displacement $d\bar{r}=[dx,dy,dz]$ everywhere in the direction of the velocity field]



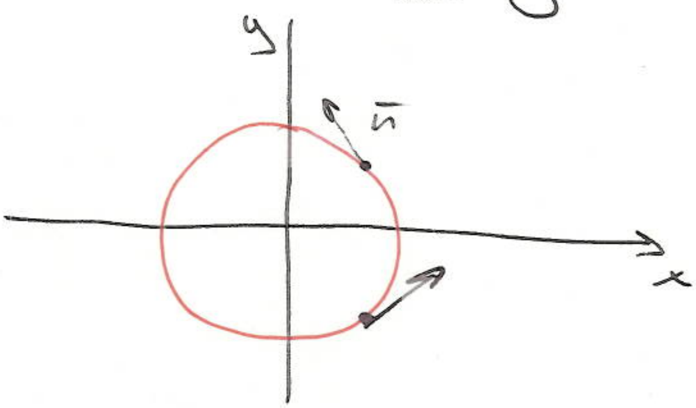
Ex (2D, steady case)

Let $u = \dfrac{-y}{x^2+y^2}$, $v = \dfrac{x}{x^2+y^2}$, $w \equiv 0$
$\Downarrow$
$dz = 0$

$x^2+y^2 > 0$

$$\frac{dx}{u} = \frac{dy}{v} \Rightarrow \frac{dx}{\frac{-y}{\cancel{x^2+y^2}}} = \frac{dy}{\frac{x}{\cancel{x^2+y^2}}}$$

$$x\,dx = -y\,dy \xrightarrow{\text{integrate}} x^2+y^2 = C$$



The time-independent trajectory is a circle

What about the motion at the origin?

In steady flows:

particle trajectories (streaklines) $\equiv$ streamlines

Not true in time-dependent flows.

$\rightarrow$ See the textbook for some more complicated examples

Remarks

$\star$ Einstein summation convention, $\bar{a} \in \mathbb{R}^N$, $\mathbb{T} \in \mathbb{R}^{N \times N}$

$$[\mathbb{T}a]_i = \sum_{j=1}^{N} T_{ij} a_j = T_{ij} a_j$$

summation on repeated indices

* Unless stated otherwise, when performing differentiation we will assume that all objects are sufficiently regular for the operation to be well defined

The Jacobian Matrix

Recall the Lagrangian flow map $M_t \bar{a} = \bar{X}(\bar{a}, t)$

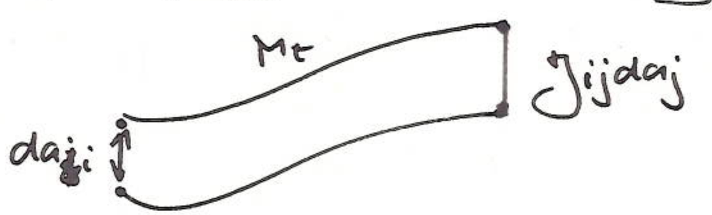
Its Jacobian is the matrix $J_{ij} = \left. \dfrac{\partial x_i}{\partial a_j} \right|_t$

Assumption

The map is one-to-one (1-1), hence invertible.

Therefore $\underline{\mathrm{Det}(J) > 0}$

$da_1 \cdots da_N$ — volume of an infinitesimal fluid parcel



$\mathrm{Det}(J)\, da_1 .. da_N$ - the parcel volume under the map M_t

Volume preserving (incompressible) flow maps:

$$\underline{\mathrm{Det}(J) = 1}$$

The Material Derivative

Suppose a particle is carrying an invariant property P (e.g., temperature)

$$\left.\frac{dP}{dt}\right|_a = 0$$ → <u>material invariance</u> (the property P remains unchanged as long as we focus on the same particle, initially at $\bar{a}$)

(wrong derivative sign in the textbook)

How to express the material invariance in the Eulerian frame of reference, i.e., at a fixed point x, rather than following a particle which started at $\bar{a}$ at $t=0$.

$$P = P(\bar{x}(\bar{a},t), t)$$

$$\left.\frac{dP(\bar{x}(\bar{a},t),t)}{dt}\right|_a = \left.\frac{\partial P}{\partial t}\right|_{\bar{x}} + \left.\frac{\partial P}{\partial x_i}\right|_t \left.\frac{\partial x_i}{\partial t}\right|_a$$

(2)

$$= \frac{\partial P}{\partial t} + \bar{u}\cdot\nabla P = \left(\frac{\partial}{\partial t} + \bar{u}\cdot\nabla\right)P$$

$\left(\frac{\partial}{\partial t} + \bar{u}\cdot\nabla\right)$ — the material / substantive / convective derivative with respect to the velocity field $\bar{u}$

Sometimes also denoted $\frac{D}{Dt}$

Ex Acceleration of a fluid particle

$$\left.\frac{d^2\bar{x}}{dt^2}\right|_a = \frac{d\bar{u}}{dt} = \underbrace{\frac{\partial\bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u}}_{\text{Eulerian}}$$

Lagrangian — Eulerian

Ex Material derivative of the determinant
(see example 1.7 for proof)

$$\frac{d}{dt}(\mathrm{Det}(J)) = (\nabla\cdot\bar{u})\,\mathrm{Det}(J),$$

where $\nabla\cdot\bar{u} = \mathrm{div}(\bar{u}) = \frac{\partial u_i}{\partial x_i}$

Thus, for volume preservation we need

$$\mathrm{Det}(J) = \text{const} \Rightarrow \underline{\nabla\cdot\bar{u} = 0}$$

* Solenoidal / incompressible / divergence-free velocity fields:

$$\underbrace{\nabla\cdot\bar{u} = 0}_{\text{Eulerian}} \iff \underbrace{\mathrm{Det}(J) = \text{const} = 1}_{\text{Lagrangian}}$$

↳ 3D: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

2D: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

* Streamfunction in 2D $\forall_t \; \psi(\cdot, t): \underline{\Omega} \rightarrow \mathbb{R}$

$$\bar{u} = [u, v] = \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right] = \nabla^{\perp} \psi$$

Note that

$$\nabla \cdot \bar{u} = \frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

Therefore

$$\nabla \cdot \bar{u} = 0 \Rightarrow \exists \psi \quad ST \quad \bar{u} = \nabla^{\perp} \psi$$

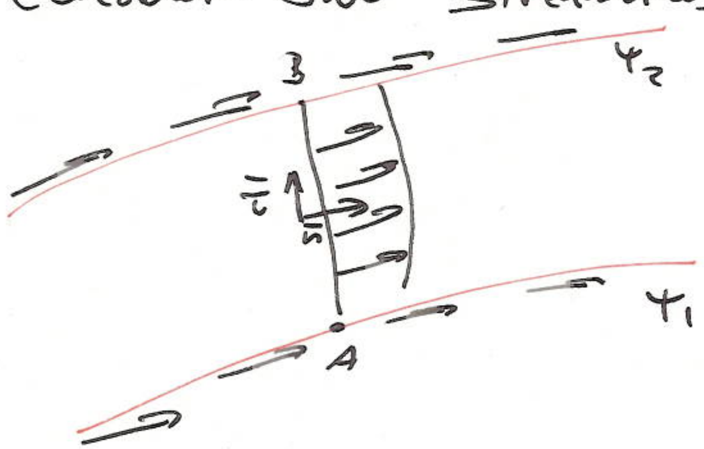
Economy of representation in 2D!

Corollary:

$0 = \nabla\psi \cdot \nabla^{\perp}\psi = \nabla\psi \cdot \bar{u} \Rightarrow$ the level sets of the streamfunction are everywhere tangent to the velocity field $\bar{u}$ (in other words, the streamfunction is constant on instantaneous streamlines)

Corollary

Consider two streamlines with streamfunction values ψ_1, ψ_2



What is the volume flux between the two streamlines?

$$\int_A^B \bar{u} \cdot \bar{n} \, ds = ?$$