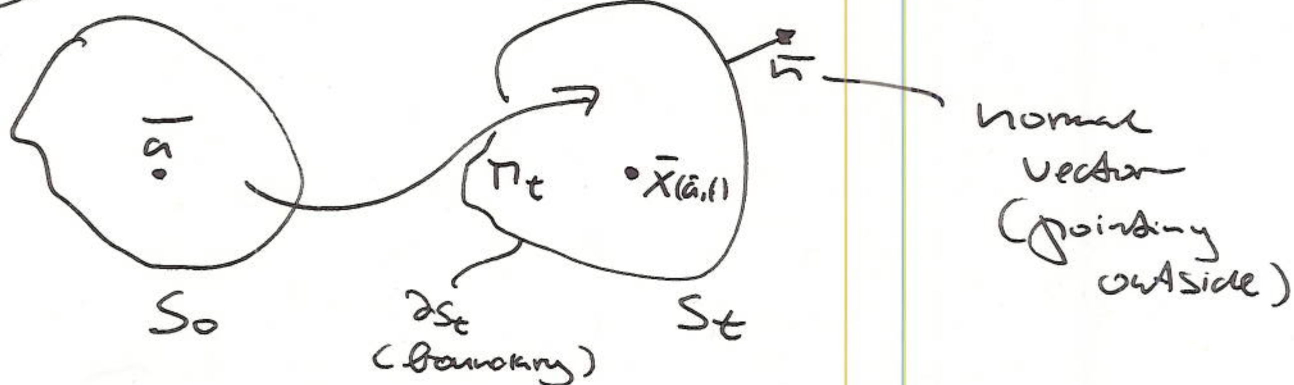
$$\int_A^B \bar{u}\cdot\bar{n}\,ds = \int_A^B \nabla^{\perp}\psi\cdot\bar{n}\,ds = \int_A^B \frac{\partial\psi}{\partial s}\,ds =$$

$$= \int_A^B d\psi = \underline{\psi_2 - \psi_1}$$

## * Reynolds Transport (convection) Theorem



Let $f$ be a scalar function defined on the fluid region $f: S_t \times [0,T] \to \mathbb{R}$

$f$ may represent some fluid property (e.g. density)

We are interested in the rate of change of

$$\int_{S_t} f(\bar{x},t)\,d\underline{\sigma}(x)$$

with time

$d\underline{\sigma}(x)$ is the integration measure associated with the fluid domain $S_t$. Note that

$$d\underline{\sigma}(\bar{x}) = dx_1 \cdots dx_N = \mathrm{Det}(J)\,da_1 \cdots da_N = \mathrm{Det}(J)\,d\underline{\sigma}_0$$

Suppose also that the flow map $M_t$ is "driven" by the velocity field $\bar{u}$.

The difficulty in differentiating $\int_{S_t} f\,d\underline{\sigma}(x)$ is that the domain $S_t$ varies with time.

Then

$$\frac{d}{dt}\int_{S_t} f(\bar{x},t)\,d\underline{o}(\bar{x}) = \frac{d}{dt}\int_{S_0} f(\bar{x}(\bar{a},t),t)\,\mathrm{Det}(J)\,d\underline{o}_0$$

$$= \int_{S_0}\left[\frac{d f(\bar{x}(\bar{a},t),t)}{dt}\,\mathrm{Det}(J) + f(\bar{x}(\bar{a},t),t)\,\frac{d}{dt}\mathrm{Det}(J)\right]d\underline{o}_0$$

$$= \int_{S_0}\left[\frac{d f(\bar{x}(\bar{a},t),t)}{dt} + f\,\nabla\cdot\bar{u}\right]\mathrm{Det}(J)\,d\underline{o}_0 = \left|\begin{array}{c}\text{transforming}\\ \text{back to}\\ \text{the fluid}\\ \text{region}\end{array}\right|$$

$$= \int_{S_t}\left[\frac{df}{dt} + f\,\nabla\cdot\bar{u}\right]d\underline{o}(\bar{x}) = \left|\begin{array}{c}\text{expanding using}\\ \text{② and product}\\ \text{rule}\end{array}\right|$$

$$= \int_{S_t}\left[\frac{\partial f}{\partial t} + u\cdot\nabla f + f\,\nabla\cdot\bar{u}\right]d\underline{o}(\bar{x}) = \int_{S_t}\left[\frac{\partial f}{\partial t} + \nabla\cdot(\bar{u}f)\right]d\underline{o}(\bar{x})$$

$$= \left|\begin{array}{c}\text{divergence}\\ \text{theorem}\end{array}\right| = \int_{S_t}\frac{\partial f}{\partial t}\,d\underline{o}(\bar{x}) + \int_{\partial S_t} f\,(\bar{u}\cdot\bar{n})\,d\sigma(\bar{x})$$

We have thus prooven the following

Theorem

$$\frac{d}{dt}\int_{S_t} f(\bar{x},t)\,d\underline{o}(\bar{x}) = \underbrace{\int_{S_t}\frac{\partial f}{\partial t}\,d\underline{o}(\bar{x})}_{\text{change within the region } S_t} + \underbrace{\int_{\partial S_t} f\,(\bar{u}\cdot\bar{n})\,d\sigma(\bar{x})}_{\text{flux accross the moving boundary}}$$

③

Combining and using the divergence theorem

$$\int_{S_t}\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{u}) - q\right] d\underline{v}(t) = 0$$

Since:

- $\rho$, $\bar{u}$ and $q$ are assumed sufficiently smooth, and
- the region $S_t$ is an arbitrary subset of $\mathbb{R}^n$

conservation of mass is expressed as

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{u}) = q \quad \text{in } S_t$$

④ OR

$$\frac{d\rho}{dt} + \rho(\nabla\cdot\bar{u}) = q \quad \text{in } S_t$$

known as the equation of continuity

Thus, the density $\rho$ may change as a result of

- production / annihilation of mass ($q$)
- compression of the fluid ($\nabla\cdot\bar{u}$)

Remark

If the fluid is incompressible ($\nabla\cdot\bar{u}=0$), and there are no sinks or sources of mass, $q\equiv 0$ then

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\bar{u}\cdot\nabla)\rho = 0 \quad \text{in } S_t$$

$(\nabla\cdot\bar{u} \neq 0) \not\Rightarrow \rho = \text{const}$. (density is conserved only along particle trajectories, and particles may initially possess different density)

However,

$$\rho = \text{const} \Rightarrow \nabla\cdot\bar{u} = 0$$

* Lagrangian form

Suppose $V$ is a volume of fluid small enough, so that the mass $\rho V$ in that volume is materially invariant

$$\frac{d}{dt}(\rho V) = 0$$

$$\left.\frac{d\rho}{dt}V + \rho\frac{dV}{dt} = 0 \Rightarrow \underbrace{\frac{d\rho}{dt} + \frac{\rho}{V}\frac{dV}{dt} = 0}_{④}\right\} \Rightarrow$$

$$\left.\begin{aligned} \nabla\cdot\bar{u} &= \frac{1}{V}\frac{dV}{dt} \\ V &= \mathrm{Det}(J)\,V_0 \end{aligned}\right\} \Rightarrow \nabla\cdot\bar{u} = \frac{1}{\mathrm{Det}(J)}\frac{d\,\mathrm{Det}(J)}{dt}$$

(for some initial volume $V_0$)

Therefore

$$\frac{d\rho}{dt} + \rho\nabla\cdot\bar{u} = \frac{d\rho}{dt} + \frac{\rho}{\mathrm{Det}(J)}\frac{d(\mathrm{Det}(J))}{dt}$$

$$= \frac{1}{\mathrm{Det}(J)}\frac{d}{dt}\left[\rho\,\mathrm{Det}(J)\right] = 0$$

## Remark

The Lagrangian form of the continuity equation is nonlinear in the flow properties $(\mathrm{Det}(\zeta))$

## Remark

Sometimes we are interested in the density $f$ per <u>unit mass</u>, rather than per unit volume, of a given conserved quantity. Then, in order to compute the amount of the quantity in the volume $S_t$ we ~~have~~ need to multiply by $\rho$,

i.e., $\int_{S_t} \rho f \, d\underline{v}(\bar{x})$

Then (assuming $q \equiv 0$)

$$\frac{d}{dt}\int_{S_t} \rho f \, d\underline{v}(\bar{x}) = \int_{S_t} \left[ \frac{\partial(\rho f)}{\partial t} + \nabla\cdot(\rho f \bar{u}) \right] d\underline{v}$$

$$= \int_{S_t} \Big[ \underbrace{f\frac{\partial \rho}{\partial t}} + \rho \frac{\partial f}{\partial t} + \underbrace{f \cancel{\rho} \nabla\cdot(\rho\bar{u})} + \rho\bar{u}\cdot\nabla f \Big] d\underline{v}$$

$= 0$ (continuity)

$$= \int_{S_t} \rho \left( \frac{\partial f}{\partial t} + \bar{u}\cdot\nabla f \right) d\underline{v} = \int_{S_t} \rho \frac{df}{dt} d\underline{v}$$

(quite different formula than <u>before</u>)

## (c) Conservation of Momentum

Momentum density (per unit volume): $\rho \overline{u}$

Newton's second law of motion: the change of momentum of a mass of fluid is equal to the sum of forces acting on it

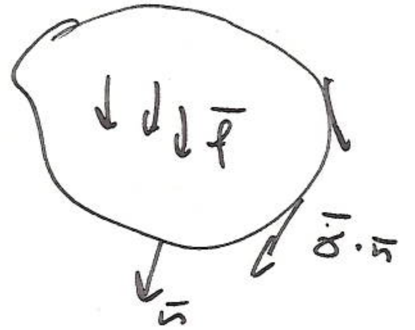
Assume: $q \equiv 0$, $\overline{F}$ – density (per unit volume) of forces

$$\left.\begin{aligned} \frac{d}{dt}\int_{S_t} \rho \overline{u}\, d\underline{x} &= \int_{S_t} \overline{F}\, d\underline{x} \\ \frac{d}{dt}\int_{S_t} \rho \overline{u}\, d\underline{x} &= \int_{S_t} \rho \frac{d\overline{u}}{dt}\, d\underline{x} \end{aligned}\right\} \Rightarrow$$

(from previous remark with $g = \overline{u}$)

$$\Rightarrow \int_{S_t} \rho \frac{d\overline{u}}{dt}\, d\underline{x} = \int_{S_t} \overline{F}\, d\underline{x} \qquad (*)$$

What about the forces $\overline{F}$? This is what distinguishes different fluid models. We need a "constitutive relation" which relates the forces (fluxes of momentum) to other state variables and fluid properties



$$\int_{S_t} \bar F\, d\underline{v} = \cancel{\int_{S_t}}$$

$$\underbrace{\int_{S_t} \bar f\, d\underline{v}}_{\text{body forces}} + \underbrace{\int_{\partial S_t} \bar\sigma\cdot\bar n\, d\sigma}_{\text{surface forces}}$$

$\bar\sigma$ – stress tensor

Body forces – e.g., gravity $\quad \bar f = \rho\bar g$

Surfaces forces – pressure, frictional (viscous) forces

We will first focus on ideal fluids in which there are no frictional forces and the only surface forces are due to pressure, $\bar\sigma\cdot\bar n = -p\bar n$

Thus $\bar\sigma = -p\underset{\substack{\uparrow \\ \text{identity} \\ \text{matrix}}}{\bar I}$, or $\sigma_{ij} = -p\delta_{ij}$

and

$$\int_{S_t} \bar F\, d\underline{v} = \int_{S_t} \rho\bar g\, d\underline{v} + \int_{\partial S_t} \bar\sigma\cdot\bar n\, d\underline{v}$$

$$= \int_{S_t} \rho\bar g\, d\underline{v} + \int_{S_t} \nabla\cdot\bar\sigma\, d\underline{v}$$

$$= \int_{S_t} \rho\bar g \mp \int_{S_t} \nabla p\, d\underline{v}$$

Finally, from ⊛ and dropping integrals

$$\int_{S_t} \varrho \frac{d\bar{u}}{dt} d\underline{x} = \int_{S_t} \varrho \bar{g}\, d\underline{x} - \int_{S_t} \nabla p \, d\underline{x}$$

⑤ $\varrho \frac{d\bar{u}}{dt} + \nabla p = \varrho \bar{g}$ in $S_t$

which together with the continuity equation

$$\frac{d\varrho}{dt} + \varrho \nabla \cdot \bar{u} = 0$$

forms Euler's equations of fluid motion.

(N+1) equations vs. (N+2) state variables

$u_1, \dots, u_N, \varrho, p$

One equation is missing...

* <u>compressible case</u> — we need an additional "equation of state" (a constitutive relation) $\varrho = \varrho(p, \dots)$ relating density to pressure and other parameters

* <u>incompressible case</u> $\varrho = \text{const} \Rightarrow \nabla \cdot \bar{u} = 0$

$\varrho$ no longer an unknown

(WLOS $\varrho \equiv 1$)

$$\begin{cases} \frac{d\bar{u}}{dt} + \nabla p = \bar{g} \\ \nabla \cdot \bar{u} = 0 \end{cases}$$

## Remark

Due to the constitutive relationship $\bar{\sigma} = -p\bar{I}$, pressure is isotropic, i.e., acts in the same way in all directions

In incompressible fluids pressure $p$ is defined up to a constant (this is not true in the compressible case).