* Local Analysis of the Velocity Field

Consider two points $\bar{x}$ and $(\bar{x}+\bar{y})$ in $\underline{\Omega}$

Expand the velocity field at $(\bar{x}+\bar{y})$ w.r.t point $\bar{x}$

$$(*)\quad u_i(\bar{x}+\bar{y},t) = u_i(\bar{x},t) + y_j \frac{\partial u_i}{\partial x_j}(\bar{x},t) + O(\|y\|^2)$$

Then decompose

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{symmetric part}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{skew-symmetric part}}$$

$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ — rate-of-strain tensor

It describes the local deformation of a fluid parcel and will play an important role later on in viscous flow models

$R_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ — rotation tensor

Note that if $\bar{\omega} = [\omega_1, \omega_2, \omega_3]^T = \nabla \times \bar{u}$

then

$$\bar{\bar{R}} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix},$$

i.e., the entries of the rotation tensor can be expressed in terms of vorticity components.

<u>Remark</u> — The rotation tensor $\bar{\bar{R}}$ does not describe the rotation "of a point", but the <u>relative</u> rotation of a point with respect to another point.

Useful identity $\quad \bar{\bar{R}}\bar{y} = +\frac{1}{2}\bar{\omega}\times\bar{y}$

Thus, Taylor expansion (*) can be rewritten as

$$\bar{u}(\bar{x}+\bar{y},t) \cong \bar{u}(\bar{x},t) + \bar{\bar{D}}(\bar{x})\bar{y} + \frac{1}{2}\,\bar{\omega}(\bar{x})\times\bar{y}$$

implying that <u>locally</u> the fluid motion can be regarded as the superposition of:

- translation $(\bar{u})$
- deformation $(\bar{\bar{D}}\bar{y})$
- rotation $(\frac{1}{2}\bar{y}\times\bar{\omega})$ with the rotation rate $\frac{1}{2}\bar{\omega}$

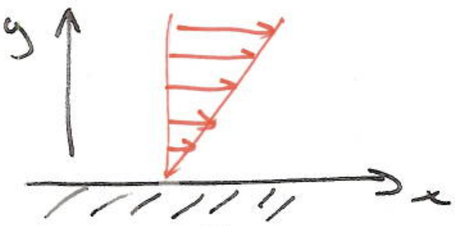
<u>Remark</u>

$$\nabla\cdot\bar{u} = \frac{\partial u_i}{\partial x_i} = D_{ii} = \mathrm{Tr}(\bar{\bar{D}})$$

$$\nabla\cdot\bar{u}=0 \iff \mathrm{Tr}(\bar{\bar{D}})=0$$

$$\mathrm{Tr}(\bar{\bar{R}})=0 \text{ by construction}$$

Ex Consider a 2D velocity field $\bar{u} = [u, v] = [y, 0]$
$x \in \mathbb{R}, y \in \mathbb{R}^+$



Trivial looking field gives rise to a nontrivial decomposition.

"shear flow"

One can show that

$$\bar{\bar{D}} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \bar{\bar{R}} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Remark

In 2D flows, vorticity becomes a "pseudo-scalar", i.e., a vector with only one non-vanishing component (the one orthogonal to the plane of motion)

$$\bar{u} = [u, v, 0]$$

$$\bar{\omega} = \nabla \times \bar{u} = \Big[0, 0, \underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{\omega_3} \Big] = \omega_3 \bar{e}_z$$

In 2D vorticity can thus be treated as a scalar.

In the above example $\omega = \omega_3 = -1$.

* ~~[illegible]~~ Circulation

Consider a simple, smooth closed oriented contour C (smooth deformation of a circle)

It bounds an oriented surface S

Consider the flux of the vorticity $\bar{\omega}$ through the surface S

$$\int_S \bar{n}\cdot\bar{\omega}\, ds = \int_S \bar{n}\cdot(\nabla\times\bar{u})\, ds \overset{\text{Stokes thm}}{=} \oint_C \bar{u}\cdot d\bar{s}$$

$$= \oint_C \bar{u}\cdot\bar{t}\, ds = \Gamma_C$$ (circulation around the contour C)

$\bar{t}$ is the unit vector tangential to the contour C.

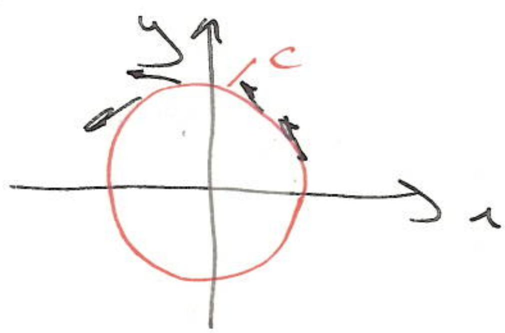
Unlike vorticity which is defined pointwise, the circulation is an integral (global) quantity.

<u>Remark</u>

Because potential flows are irrotational ($\nabla\times\nabla\phi\equiv 0$), their circulation is always zero.

<u>Ex</u> (point vortex)

Consider 2D velocity field $\bar{u} = [u, v] = \frac{1}{2\pi}\left[-\frac{y}{r^2}, \frac{x}{r^2}\right]$



As C consider a circle of radius r_0

$$C = \left\{ (x,y)\in\mathbb{R}^2,\ x^2+y^2=r_0^2 \right\}$$

$$\Gamma_C = \oint_C \bar{u}\cdot\bar{t}\, ds \underset{\substack{\uparrow \\ \text{switch to the} \\ \text{polar coordinate} \\ \text{system}}}{=} \int_0^{2\pi} u_\theta\, r\, d\theta = r_0 \int_0^{2\pi} \frac{1}{2\pi r_0}\, d\theta = \frac{2\pi r_0}{2\pi r_0} = 1$$

$$\bar{u} = u\bar{e}_x + v\bar{e}_y = u_r\bar{e}_r + u_\theta\bar{e}_\theta$$

$$u_r = 0,\quad u_\theta = \frac{1}{2\pi r}$$

Thus, the circulation Γ_C of the point vortex is one and does not depend on the choice of the contour C.

* Kelvin's Theorem

Consider a restricted class of compressible fluids known as the barotropic fluid. For such fluids the ~~pres~~ pressure is only a function of the density, i.e., $p = p(\rho)$ (but not of the temperature). The continuity and momentum equations form a closed system

~~Incompressible fluids are a special ...~~

Theorem (~~Lo~~ Kelvin)

Suppose $C(t)$ is a simple closed material curve moving in an ideal fluid subject to the body force $\bar{f} = -\rho \nabla \Phi$. Then, if

- the fluid is incompressible ($\rho = \text{const}$), or
- barotropic

the circulation $\Gamma_{C(t)}$ of the velocity field $\bar{u}$ on $C(t)$ is invariant under the flow, i.e.,

(8) $$\frac{d}{dt} \Gamma_{C(t)} = 0$$

Proof

Consider parametrisation of $C(t)$ in terms of Lagrangian particle trajectories

$$C(t) = \left\{ \bar{x}(\bar{\alpha}, t),\ \alpha \in [0, L] \right\} \qquad \frac{d\Gamma_{C(t)}}{dt} =$$

$$= \frac{d}{dt} \oint_C \bar{u} \cdot \bar{t}\, ds = \frac{d}{dt} \int_0^L \bar{u} \frac{\partial \bar{x}}{\partial \alpha} d\alpha = \int_0^L \left(\frac{d\bar{u}}{dt} \cdot \frac{\partial \bar{x}}{\partial \alpha} + \bar{u} \cdot \frac{\partial \bar{u}}{\partial \alpha} \right) d\alpha$$

Now use the momentum equation

$$\frac{d\bar{u}}{dt} + \frac{1}{\rho} \nabla p + \nabla \underline{\phi} = 0$$

$$= \int_0^L \left[-\left(\frac{1}{\rho} \nabla p + \nabla \underline{\phi} \right) \frac{\partial \bar{x}}{\partial \alpha} + \bar{u} \cdot \frac{\partial \bar{u}}{\partial \alpha} \right] d\alpha =$$

$$= \left| \begin{matrix} \nabla_{\bar{x}} p \frac{\partial \bar{x}}{\partial \alpha} d\alpha = dp \\ \ldots \end{matrix} \right| = \oint_C \underbrace{\left[-\frac{dp}{\rho} + d\left(\frac{1}{2} |\bar{u}|^2 + \underline{\phi} \right) \right]}_{\text{complete differential along a closed curve of a single-valued function.}} = 0$$

Kelvin's theorem expresses a very important Lagrangian property of the ~~...~~ ideal flows.

* The Vorticity Equation

Assume the fluid is:

- barotropic ($p = p(\rho)$), or

- incompressible ($\rho = \text{const.}$)

To obtain the vorticity equation we start by taking the curl of the momentum equation

$$\nabla \times \Big(\underbrace{\frac{d\bar{u}}{dt}}_{\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u}} + \frac{1}{\rho}\nabla p + \nabla \bar{\phi} \Big) = 0$$

$$\nabla \times \frac{d\bar{u}}{dt} = \nabla \times \frac{\partial \bar{u}}{\partial t} + \nabla \times \left[(\bar{u}\cdot\nabla)\bar{u} \right] = 0$$

$$\nabla \times \frac{\partial \bar{u}}{\partial t} = \frac{\partial}{\partial t}(\nabla \times \bar{u}) = \frac{\partial \bar{\omega}}{\partial t}$$

For the nonlinear term use Lamb's identity

$$(\bar{u}\cdot\nabla)\bar{u} = \frac{1}{2}\nabla|\bar{u}|^2 - \bar{u}\times\bar{\omega}$$

so that

$$\nabla \times \left[(\bar{u}\cdot\nabla)\bar{u}\right] = \frac{1}{2}\underbrace{\nabla\times\nabla|\bar{u}|^2}_{0} - \nabla\times(\bar{u}\times\bar{\omega})$$

As regards the second term, we have the following vector identity ($\bar{A}, \bar{B} : \underline{\Omega} \to \mathbb{R}^N$)

$$\nabla \times (\bar{A} \times \bar{B}) = (\bar{B}\cdot\nabla)\bar{A} - (\bar{A}\cdot\nabla)\bar{B} + \bar{A}(\nabla\cdot\bar{B}) - \bar{B}(\nabla\cdot\bar{A})$$

We always have $\nabla \cdot \bar{\omega} = 0$

* In the incompressible case ($\rho = $ const) we also have $\nabla \cdot \bar{u} = 0$

Then, after rearranging terms

$$(9) \quad \frac{\partial \bar{\omega}}{\partial t} + (\bar{u} \cdot \nabla)\bar{\omega} = \frac{d\bar{\omega}}{dt} = \bar{\omega} \cdot \nabla \bar{u}$$

* in the compressible / barotropic case we have instead (using conservation of mass) $\nabla \cdot \bar{u} = -\frac{1}{\rho}\frac{d\rho}{dt}$ so that the vorticity equation becomes

$$(10) \quad \frac{d\bar{\omega}}{dt} = \bar{\omega} \cdot \nabla \bar{u} + \frac{\bar{\omega}}{\rho}\frac{d\rho}{dt}$$

After rearrangement

$$\frac{d}{dt}\left(\frac{\bar{\omega}}{\rho}\right) = \left(\frac{\bar{\omega}}{\rho}\right) \cdot \nabla \bar{u}$$

Remarks

* when $\rho = $ const, (10) becomes (9)

* equations (9) – (10) show that as $\bar{\omega}$ or $\frac{\bar{\omega}}{\rho}$ moves with the fluid (LHS), it is stretched by the velocity field (RHS)

"Vortex-stretching" term: $(\bar{\omega} \cdot \nabla \bar{u})_i = \omega_j \frac{\partial u_i}{\partial x_j}$

* in 2D $\quad \bar{\omega} \cdot \nabla \bar{u} = (\omega \bar{e}_z) \cdot \nabla \bar{u} = 0$ and

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + (\omega \cdot \nabla)\bar{u} = 0$$

Vorticity is a <u>material invariant</u> and cannot change inside the domain (it may only be produced on the boundary)

* Lagrangian Form of the Vorticity Equation (incompressible case)

$$\omega_i(\bar{a}, t) = \underbrace{\mathcal{J}_{ij}(\bar{a}, t)}_{} \, \omega_{0j}$$

$\bar{\omega}_0$ - initial vorticity

connect the vorticity evolution with the deformation of the flow