* in 2D $\quad \bar{\omega}\cdot\nabla\bar{u} = (\omega\bar{e}_z)\cdot\nabla\bar{u} = 0 \quad$ and

$$\frac{d\omega}{dt} = \frac{\partial\omega}{\partial t} + (\omega\cdot\nabla)\bar{u} = 0$$

Vorticity is a <u>material invariant</u> and cannot change inside the domain (it may only be produced on the boundary)

* Lagrangian Form of the Vorticity Equation (incompressible case)

$$\omega_i(\bar{a},t) = \mathcal{J}_{ij}(\bar{a},t)\,\omega_{0j}$$

$\bar{\omega}_0$ - initial vorticity

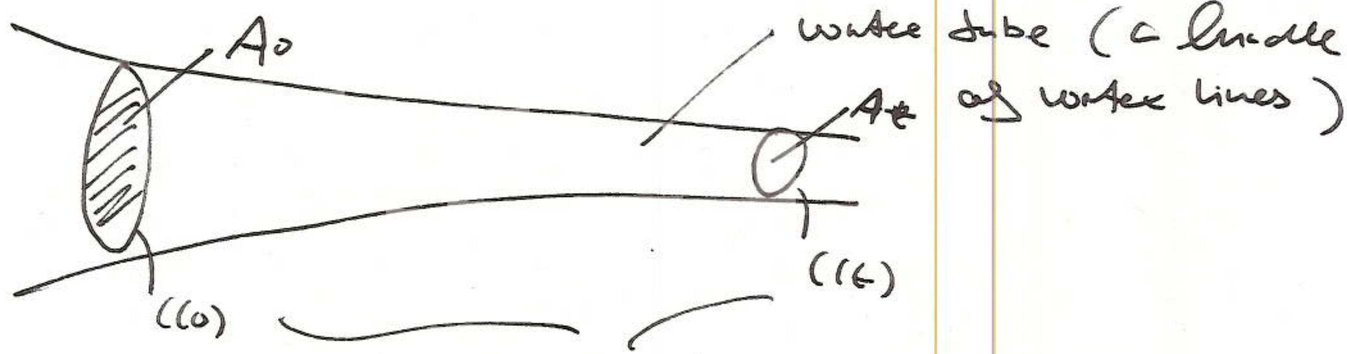
connect the vorticity evolution with the deformation of the flow

* <u>Helmholtz' Laws of Vortex Motion</u> (derived in mid-19th century, are consequences of Kelvin's principle and the vorticity equation

~~In~~ In an <u>ideal</u> fluid

- fluid particles free of vorticity stay free of vorticity
- vortex lines are material lines
- the strength of the vortex tube is an invariant of motion.

Regarding the third law,



Let A_0, A_t be the areas of the surface bounded by $C(0)$ and $C(t)$. The contour $C(t)$ moves with the flow field.

To fix attention, assume that $\omega_0 = \text{const}$
$\omega_t = \text{const}$

Then, from Kelvin's principle,

$$\Gamma_C = \omega_0 A_0 = \omega_t A_t \Rightarrow \omega_t = \omega_0 \frac{A_0}{A_t}$$

<u>Vortex stretching</u> mechanism: as the vortex tube is stretched by the flow, $A_t \downarrow$ and $\omega_t \nearrow$. This phenomenon is observed in actual turbulent flows.

<u>Open Question</u>: can it happen that $\omega_t \to \infty$? (in equations, in reality this cannot of course happen)

* <u>How to obtain the velocity field from a given vorticity field?</u>

Assume $\underline{O} = \mathbb{R}^3$

Vorticity $\bar{\omega} : \underline{O} \to \mathbb{R}^3$ is given and assumed to decay sufficiently rapidly at infinity. More specifically,

$$|\omega| < C r^{-N}, \quad \text{as } r \to \infty$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

Then $\nabla \times \bar{v} = \nabla \times (\bar{u} + \nabla \varphi) = \bar{\omega}$

nonuniqueness relates to the presence of the potential field

Assume also that $\lim_{|x| \to \infty} |v(x)| = 0$

Take one more curl:

$$\nabla \times \nabla \times \bar{u} = \nabla \times \bar{\omega}$$

Use the identity $\nabla \times \nabla \times \bar{u} = \nabla (\underbrace{\nabla \cdot \bar{u}}_{0}) - \Delta \bar{u}$ $\Bigg\} \Rightarrow$

(our field is incompressible)

$\Delta \bar{u} = - \nabla \times \bar{\omega}$ – vector Poisson equation for $\bar{u}$ in $\mathbb{R}^3$

The <u>fundamental</u> solution (Newton's potential) is

$$G(\bar{x},\bar{y}) = \frac{1}{|\bar{x}-\bar{y}|}\frac{1}{4\pi}, \quad \bar{x}\neq\bar{y}$$

Then, the solution $\bar{u}$ is given by

$$\bar{u}(\bar{x}) = \frac{-1}{4\pi}\int_{\underline{\Omega}}\frac{\nabla_y \times \omega(\bar{y})}{|\bar{x}-\bar{y}|}\,d\underline{\Omega}(\bar{y}) = \left|\begin{array}{l}\underline{\Omega}\approx\underline{\Omega}_0 \\ \text{assume } \text{is finite} \\ (\text{meas}(\underline{\Omega}_0)<\infty)\ \& \\ \text{integrate by} \\ \text{parts}\end{array}\right|$$

$$= \frac{1}{4\pi}\int_{\underline{\Omega}_0}\nabla_y\left(\frac{1}{|\bar{x}-\bar{y}|}\right)\times\bar{\omega}(\bar{y})\,d\underline{\Omega}(\bar{y}) + \frac{1}{4\pi}\oint_{\partial\underline{\Omega}_0}\frac{\bar{n}\times\bar{\omega}(\bar{y})}{|\bar{x}-\bar{y}|}\,d\sigma(\bar{y})$$

$$= \left|\begin{array}{l}\text{taking } \partial\underline{\Omega}_0\to\infty \text{ the second integral vanishes} \\ \text{due to the assumed decay of vorticity.} \\ \text{Moreover, due to the symmetry of the} \\ \text{fundamental solution}\end{array}\right|$$

$$-\nabla_y\left(\frac{1}{|\bar{x}-\bar{y}|}\right) = \nabla_x\left(\frac{1}{|\bar{x}-\bar{y}|}\right)$$

$$= -\frac{1}{4\pi}\int_{\underline{\Omega}}\nabla_x\left(\frac{1}{|\bar{x}-\bar{y}|}\right)\times\bar{\omega}(\bar{y})\,d\underline{\Omega}(\bar{y})$$

$$= \frac{1}{4\pi}\int_{\underline{\Omega}}\frac{(\bar{y}-\bar{x})\times\bar{\omega}(\bar{y})}{|\bar{x}-\bar{y}|^{3}}\,d\underline{\Omega} = \underbrace{\int_{\underline{\Omega}}\bar{K}(\bar{x},\bar{y})\,\bar{\omega}(\bar{y})\,d\underline{\Omega}(\bar{y})}_{\substack{\text{Convolution-type} \\ \text{integral} \to \\ \text{Biot-Savart law} \\ = u(\bar{x})}}$$

Biot-Savart Kernel:

$$\bar{K}(\bar{x},\bar{y}) = \frac{1}{4\pi}\frac{(\bar{y}-\bar{x})\times(\cdot)}{|\bar{x}-\bar{y}|^{3}}$$

Thus, $\bar{v}(x) = \int_{\Omega} \bar{K}(\bar{x},\bar{y})\bar{\omega}(\bar{y})\,d\Omega(\bar{y}) + \nabla\varphi$ (11)

What about the potential part?

Since

$$\lim_{|x|\to\infty} \left[\int_{\Omega} \bar{K}(\bar{x},\bar{y})\bar{\omega}(\bar{y})\,d\Omega(\bar{y})\right] = 0$$

we ~~...~~ must have $\varphi = C$ as the only smooth function satisfying $\lim_{|x|\to\infty} |\nabla\varphi(x)| = 0$. WLOG, we may set $\varphi \equiv 0$.

The potential φ would be nontrivial on a domain with solid boundaries and some prescribed BCs.

Remark

The form of the fundamental solution $G(\bar{x},\bar{y})$ of the Laplace/Poisson equation depends on the spatial dimension

E.g., in 2D

$$G(\bar{x},\bar{y}) = -\frac{1}{2\pi}\ln|\bar{x}-\bar{y}|$$

(logarithmic potential)

Remark

If the divergence $q = \nabla\cdot\bar{u}$ of the velocity field $\bar{u}$ is given, than the velocity field $\bar{u}$ can be reconstructed in a similar way as above. It will be defined up to a solenoidal ~~...~~ field.

Digression – momentum equation without pressure

Consider equations for an incompressible fluid ($\rho=1$)

$$\left.\begin{aligned} &\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u} + \nabla p = 0 \\ &\nabla\cdot\bar{u} = 0 \end{aligned}\right\} \quad \text{in } (0,T]\times\underline{\Omega}$$

$+$ BC, IC

Take the divergence of the momentum equation

$$-\nabla\cdot\nabla p = -\Delta p = \nabla\cdot[(\bar{u}\cdot\nabla)\bar{u}]$$

So that

$$p = -\Delta^{-1}\nabla\cdot[(\bar{u}\cdot\nabla)\bar{u}]$$

where the inverse Laplacian Δ^{-1} incorporates suitable BCs for pressure.

Plugging this back to the momentum equation

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u} - \nabla\Delta^{-1}\nabla\cdot[(\bar{u}\cdot\nabla)\bar{u}] = 0$$

$$\frac{\partial \bar{u}}{\partial t} + \underbrace{\left[Id - \nabla\Delta^{-1}\nabla\cdot\right]}_{P}(\bar{u}\cdot\nabla)\bar{u} = 0$$

$$\frac{\partial \bar{u}}{\partial t} + P[(\bar{u}\cdot\nabla)\bar{u}] = 0 \qquad (12)$$

P – Leray projection (from L_2 onto its divergence-free subspace

$P: L_2(\underline{\Omega}) \to S$

$S: \{\bar{v}\in L_2(\underline{\Omega}),\ \nabla\cdot\bar{u} = 0\}$

It can be expressed explicitly as (when $\underline{O} = \mathbb{R}^N$, i.e., when there are no solid boundaries)

$$P(\bar{v})(\bar{x}) = \bar{v}(\bar{x}) - \nabla \int_{\underline{O}} G(\bar{x}, \bar{y}) \, (\nabla \cdot \bar{v}(\bar{y})) \, d\underline{O}(\bar{y})$$

* Boundary Conditions for the Vorticity Equation

Assume the flow domain $\underline{O}$ is bounded by solid boundaries $\partial \underline{O}$. The only BC we have is for the velocity

$$(*) \qquad \bar{u} \cdot \bar{n} = u_b^n \quad \text{on } \partial \underline{O}$$

How do convert it into a BC on $\bar{\omega}$?

Given $u_b^n : \partial \underline{O} \to \mathbb{R}$ only, we cannot compute $\nabla \times \bar{u}$ on boundary

We can use condition $(*)$ combined with the Biot-Savart law to obtain

$$\bar{n} \cdot \left[\int_{\underline{O}} \bar{K}(\bar{x}, \bar{y}) \bar{\omega}(\bar{y}) d\underline{O}(\bar{y}) + \nabla \varphi \right] = u_b^n \quad \text{on } \partial \underline{O}$$

which is a nonlocal boundary condition.

The fact that the BCs for the vorticity equation are
- nonlocal
- expressed in terms of vorticity

is a major complication

* Some Analytical Examples of Vortex Flows

We have already seen the "point vortex"

Assume:

- the flow is in 2D
- the flow is steady $\frac{\partial}{\partial t} \equiv 0$

Then, since $\frac{\partial \omega}{\partial t} = 0$, the vorticity equation becomes
$$(\bar{u} \cdot \nabla)\omega = 0$$
Introduce streamfunction ψ, so that $\bar{u} = \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right]$
and the vorticity becomes
$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = \Delta \psi$$
Then
$$(\bar{u} \cdot \nabla)\omega = \underbrace{\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\Delta\psi - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\Delta\psi}_{J(\Delta\psi,\, \psi)} = 0$$