$J(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$ is the

<u>Jacobian determinant</u>

It has the following property

If $J(f,g) = 0 \iff f$ and g a functionally related,

i.e., $f = F(g)$

In our case, therefore,

$$\Delta\psi = F(\psi) \quad \text{in } \underline{\Omega}, \qquad (13)$$

where $F : \mathbb{R} \to \mathbb{R}$ must be sufficiently smooth, but is otherwise undetermined

Thus, in 2D steady Euler flows

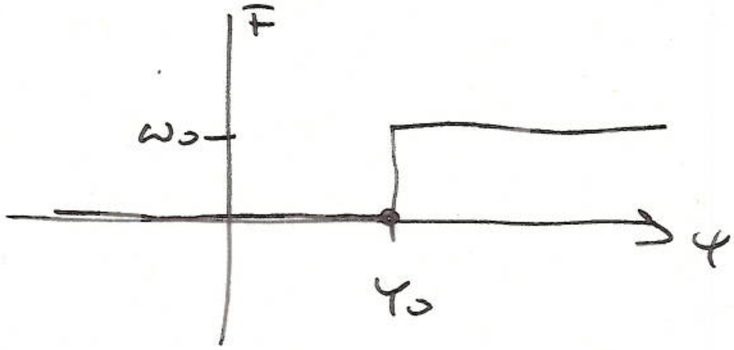
- vorticity is always a function of the streamfunction (constant on streamlines)
- <u>nonuniqueness</u> related to the arbitrariness of F

<u>Examples</u>

Assume

- the flow domain is unbounded $\underline{\Omega} = \mathbb{R}^2$
- solutions are axisymmetric, $\frac{\partial}{\partial\theta} \equiv 0$, $\psi = \psi(r)$

Let $F(\psi) = \omega_0 [1 - H(\psi_0 - \psi)]$, $H(\cdot)$ - Heaviside (step) function



ω_0, ψ_0 - given

Eq. (B) becomes in the polar coordinates

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = \omega_0\left[1 - H(\psi_0 - \psi(r))\right]$$

which is a simple linear ODE and can be solved

Expressing the solution in terms of the azimuthal velocity $u_\theta = -\frac{\partial \psi}{\partial r}$ we have

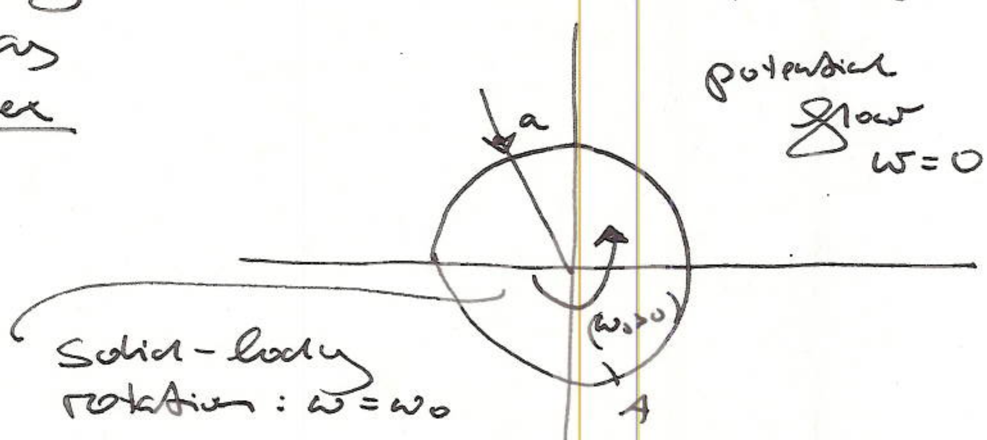
$$u_\theta = \cancel{\frac{\omega_0 a^2}{r}} \begin{cases} \dfrac{\omega_0 a^2}{r}, & r \geq a \\ \omega_0 r, & r < 0 \end{cases}$$

where a is defined such that $\psi(a) = \psi_0$

This is known as the <u>Rankine Vortex</u>

The circulation is

$$\Gamma = \int_A \omega \, dA = \omega_0 \pi a^2$$



If we fix $\Gamma = \pi \omega_0 a^2 = \text{const}$ and let $a \to 0$, then $\omega_0 \to \pm\infty$ and we obtain the point vortex

Ex

Eliminate the assumption $\frac{\partial}{\partial\theta} \neq 0$

Assume $F(\psi) = -k^2\psi$

Using the ansatz $\psi(r,\theta) = h(r)\sin\theta$, the equation

$$\Delta\psi + k^2\psi = 0$$

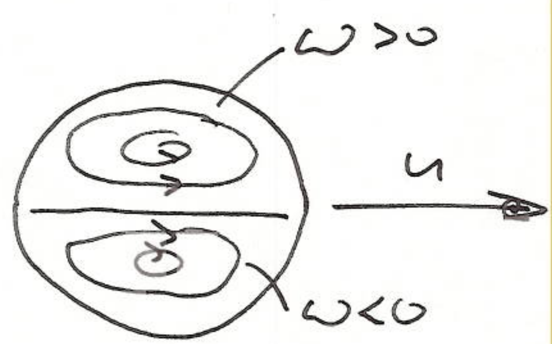
transforms into a Bessel-type ODE for $h(r)$

The solution is therefore $h(r) = C J_1(kr)$, J_1 - Bessel function of order 1

$$\left.\begin{aligned} \psi(r,\theta) &= C\sin\theta\, J_1(kr) \\ \omega(r,\theta) &= C k^2 \sin\theta\, J_1(kr) \end{aligned}\right\} r \le a \quad \text{(inside the vortex core)}$$

The constants C and k are fixed to ensure continuity of ψ and ω in the flow domain.

The solution is a propagating vortex dipole

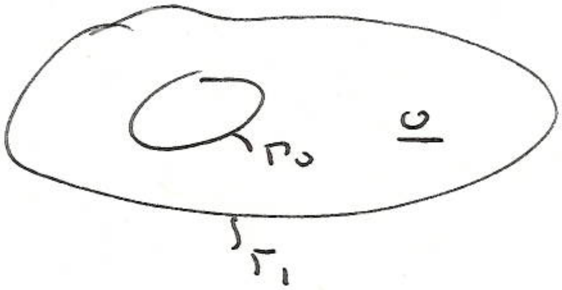


Steady flow in the moving frame of reference

Analogous constructions exist in the 3D axisymmetric (steady) case

* More on Potential Flows

Consider a 2D annular region $\underline{o}$



$$\partial \underline{o} = \Gamma_0 \cup \Gamma_1$$

$$\bar{u} = \nabla \phi$$

Incompressible flows

$$\left.\begin{aligned} \nabla \cdot \bar{u} = \Delta \phi &= 0 && \text{in } \underline{o} \\ \frac{\partial \phi}{\partial n} &= u_b^n && \text{on } \partial \underline{o} \end{aligned}\right\} (*)$$

Functions ϕ satisfying Laplace's equation are called "harmonic"

$\Rightarrow$ harmonic flows

The theory of such functions is well developed (dating back to pre-computer ~~era~~ era), but there are some subtle points.

Are solutions to problem (*) unique?

Suppose $\bar{u}_1 = \nabla \phi_1$, $\bar{u}_2 = \nabla \phi_2$ are two hypothetical solutions of problem (*).

Denote $\tilde{u} = \bar{u}_2 - \bar{u}_1 = \nabla \overbrace{(\phi_2 - \phi_1)}^{\tilde{\phi}} = \nabla \tilde{\phi}$

If sol'n are unique, then $\bar{u}_1 \equiv \bar{u}_2$ and $\tilde{u} \equiv 0$

We ~~then~~ should then have

$$\int_{\underline{o}} \tilde{u}^2 \, d\underline{o} = 0$$

Let's see...

$$\int_{\underline{O}} \tilde{u}^2 \, d\underline{O} = \int_{\underline{O}} (\nabla\tilde{\phi})^2 \, d\underline{O} = \int_{\underline{O}} \Big[(\nabla\tilde{\phi})^2 + \tilde{\phi}\underbrace{\Delta\tilde{\phi}}_{0}\Big] d\underline{O}$$

$$= \int_{\underline{O}} \nabla\cdot\big[\tilde{\phi}\nabla\tilde{\phi}\big] \, d\underline{O} = \left|\begin{array}{c}\text{divergence}\\ \text{theorem}\end{array}\right| = \oint_{\partial\underline{O}} \tilde{\phi}\frac{\partial\tilde{\phi}}{\partial n} d\sigma$$

$$= 0 \quad \underline{\text{OK}}$$

But the divergence theorem applies <u>only</u> when $\tilde{\phi}$ is <u>single-valued</u> in $\underline{O}$. Is it?

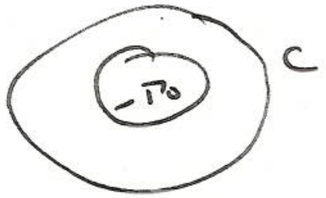
What does this mean?

single-valued $\equiv$ $\oint_C d\phi = 0$ along <u>any</u> contour C in $\underline{O}$

If domain $\underline{O}$ is <u>simply connected</u>, ~~then~~ contour C can be shrunk to a point and $\oint_C d\phi = 0$ is satisfied trivially

<u>Remark</u>

* the exterior of a sphere in 3D is simply connected
* the exterior of a disk in 2D is <u>not</u> simply connected



Conclusion

The potential ϕ defined in 2D annular domain need not be single-valued. Thus, in such case

$\int_{\Omega} (\nabla \tilde{\phi})^2 d\Omega$ need not vanish and problem (*) may have nonunique solutions.

$$\oint_C d\phi = \oint_C \frac{\partial \phi}{\partial s} ds = \oint_C \bar{V} \cdot \bar{t}\, ds = \Gamma_C$$

To make the solutions unique, one must complement system (*) by prescribing the circulation Γ_C.

If there is circulation, there must be vorticity (Stokes thm), where is it?

Ex

Consider the flow in an annular domain ~~~~ between two concentric cylinders with radii R_0 and R_1

$$\bar{u} = \frac{\Gamma}{2\pi}\left(-\frac{y}{r^2}\bar{e}_x + \frac{x}{r^2}\bar{e}_y\right) \quad \text{(Cartesian)}$$

$$= \frac{\Gamma}{2\pi}\left(\frac{1}{r}\bar{e}_\theta + 0\,\bar{e}_r\right) \quad \text{(polar)}$$

$$\bar{u} = \nabla\phi = \frac{1}{r}\frac{\partial \phi}{\partial \theta}\bar{e}_\theta + \frac{\partial \phi}{\partial r}\bar{e}_r$$

– point vortex with strength (circulation) Γ

Thus:

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r} \implies \phi(r,\theta) = \frac{\Gamma}{2\pi}\Theta + C$$

which is clearly multivalued

Different flows are obtained for different values of Γ (the circulation must be fixed to obtain a unique solution)

* Harmonic Flows in 2D & Complex Variables

There is a deep connection between harmonic flows and the theory of the functions of complex variables

Assume $\left.\begin{array}{l}\nabla\cdot\bar{u}=0\\ \nabla\times\bar{u}=0\end{array}\right\}$ almost everywhere in $\underline{\Omega}$

Let $z = x+iy$

Define the complex potential $w : \underline{\Omega} \to \mathbb{C}$

$$w(z) = \underbrace{\phi(x,y)}_{\text{potential}} + i\underbrace{\psi(x,y)}_{\text{stream function}} \qquad i=\sqrt{-1}$$

Note that the velocity components can be expressed as

$$[u,v] = \left[\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right] = \left[\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right]$$

So that $\underbrace{\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}}$

Cauchy-Riemann conditions

Together with some regularity assumptions, these conditions imply that the complex potential is holomorphic (complex differentiable) and analytic.

Thus:

(14) $$\frac{dw}{dz} = V(z) = u(x+iy) - i\,v(x+iy)$$

Remark

The Cauchy-Riemann conditions imply that

$$\nabla \phi \cdot \nabla \psi = 0$$ wherever the gradients are defined

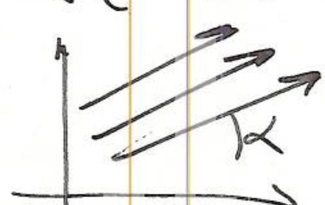
meaning that the streamfunctions are everywhere orthogonal to the lines of constant potential.

Ex (uniform flow)

$$W(z) = Q z e^{-i\alpha}$$

$$V(z) = \frac{dW}{dz} = Q e^{-i\alpha} = Q(\cos\alpha - i\sin\alpha)$$

$$\Rightarrow \begin{array}{l} u = Q\cos\alpha \\ v = Q\sin\alpha \end{array}$$



Ex (point vortex)

$$W(z) = \frac{\Gamma}{2\pi i} \ln z = -\frac{i\Gamma}{2\pi} \ln z$$

$$V(z) = \frac{dW}{dz} = \frac{\Gamma}{2\pi i z} = \underbrace{\frac{-\Gamma}{2\pi}\left(\frac{y}{r^2} + i\frac{x}{r^2}\right)}_{\text{known expression for the velocity components of a point vortex}}$$

Moreover

$$W(z) = \frac{-i\Gamma}{2\pi} \ln z = \frac{-i\Gamma}{2\pi}\left(\underbrace{\ln|z| + i(\theta + 2\pi n)}_{\text{take the principal branch } (n=0)\ (\text{multivalued function})}\right)$$

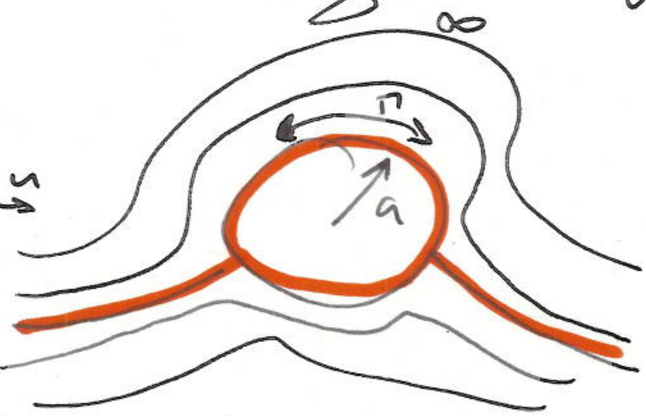
$$= \underbrace{\frac{\Gamma}{2\pi}\theta}_{\text{potential}} + i\underbrace{\frac{\Gamma}{2\pi}\ln r}_{\text{stream function}} \qquad (r = |z|)$$

(fundamental sol'n of Laplace eq. in 2D)

Ex (flow past a rotating cylinder)

$$W(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi} \ln z$$

uniform flow at ∞ — dipole — point vortex



$$\frac{dW}{dz} = u - iv$$

One can check that

$[u, v] \cdot \vec{n} = 0$ on the boundary