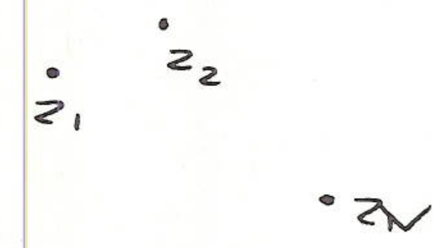
* N-vortex problem

Consider N point vortices with complex coordinates



$$z_k(t) = x_k(t) + i y_k(t)$$

$$k = 1, \dots, N$$

The complex velocity induced by k-th vortex is

$$V(z) = (u - iv) = \frac{-i}{2\pi} \frac{\Gamma_k}{z - z_k}$$

The evolution of the ~~coupled system of ODEs~~ is ~~therefore given by~~ vortex system is therefore given by the coupled system of ODEs

$$(15) \quad \frac{d\bar{z}_j}{dt} = -\frac{i}{2\pi} \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{\Gamma_k}{z_j - z_k} = -\frac{i}{2\pi} {\sum_{k=1}^{N}}' \frac{\Gamma_k}{z_j - z_k}$$

$$j = 1, \dots, N$$

where the singular "self-induction" terms has been excluded (there are actually mathematical reasons for doing that)

Remarks

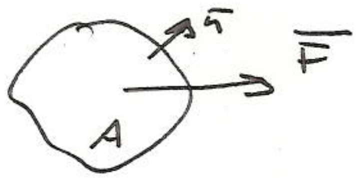
* System (15) enjoys a number of very interesting mathematical properties (e.g., it has Hamiltonian structure, etc.)
* System (15) has similarities to the gravitational N-body problem encountered in astrophysics

Representations of 2D potential flows can be used in combination with the conformal mapping theory (in terms of complex variables) to determine flows in complicated domains.

On-line examples:
* conformal mapping
* potential flow machine

Aerodynamics - the study of flows past objects (cars, planes, etc.)

Key object of interest - aerodynamic force



$$\bar{F} = -\int_{\partial A} p\,\bar{n}\,d\sigma \quad \text{(in ideal fluid)}$$

drag - component of the force colinear with the direction of motion

lift - component of the force orthogonal to the direction of motion

Blasius' Theorem: Let a steady uniform flow past a fixed 2D object with bounding contour C

Be a harmonic flow with the velocity potential $w(z)$. Then, if no external body forces are present, the force $\bar{F} = F_x \bar{e}_x + F_y \bar{e}_y$ exerted by the fluid on the body is given by

$$F_x - iF_y = \frac{i\rho}{2} \oint_C \left(\frac{dw}{dz}\right)^2 dz \qquad (16)$$

C

Proof

Consider an infinitesimal element (dx, dy) of the contour. The force acting on it is

$$dF_x - i\,dF_y = p(-dy - i\,dx) = -ip\,d\bar{z}$$

From Bernoulli's equation: $p = -\frac{\rho}{2}\left|\frac{dw}{dz}\right|^2 + C$

Thus

$$F_x - iF_y = -i \oint_C p\,d\bar{z} = \frac{i\rho}{2} \oint_C \left|\frac{dw}{dz}\right|^2 d\bar{z} = \frac{i\rho}{2} \oint_C \frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}\,d\bar{z}$$

Contour C is a streamline: $\psi = \text{Const} \Rightarrow d\psi = 0$

Therefore $\frac{d\bar{w}}{d\bar{z}}\,d\bar{z} = \overline{\frac{dw}{dz}\,dz} = \overline{dw} = dw = \frac{dw}{dz}\,dz$

So that

$$F_x - iF_y = \frac{i\rho}{2} \oint_C \left(\frac{dw}{dz}\right)^2 dz \qquad \blacksquare$$

Let's return to the flow past (rotating) cylinder

$$W(z) = U\left(z+\frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z$$

The flow has a fore-and-aft symmetry regardless of the circulation Γ $\Rightarrow$ fluid momentum doesn't change in x.

~~More generally we have~~

~~D'Alembert's Paradox~~

Using Blasius formula with the potential $W(z)$ given above, we obtain

$$F_x - iF_y = i\rho\Gamma U$$

Remarks

* drag vanishes: $F_x = 0$!

More generally, we have

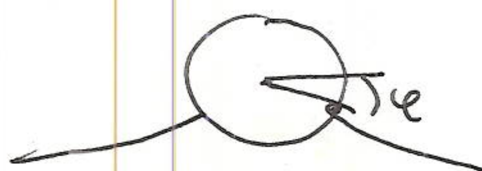
D'Alembert's Paradox

In a steady flow of an ideal fluid past an object of finite extent approaching uniform flow at large distances the drag is zero.

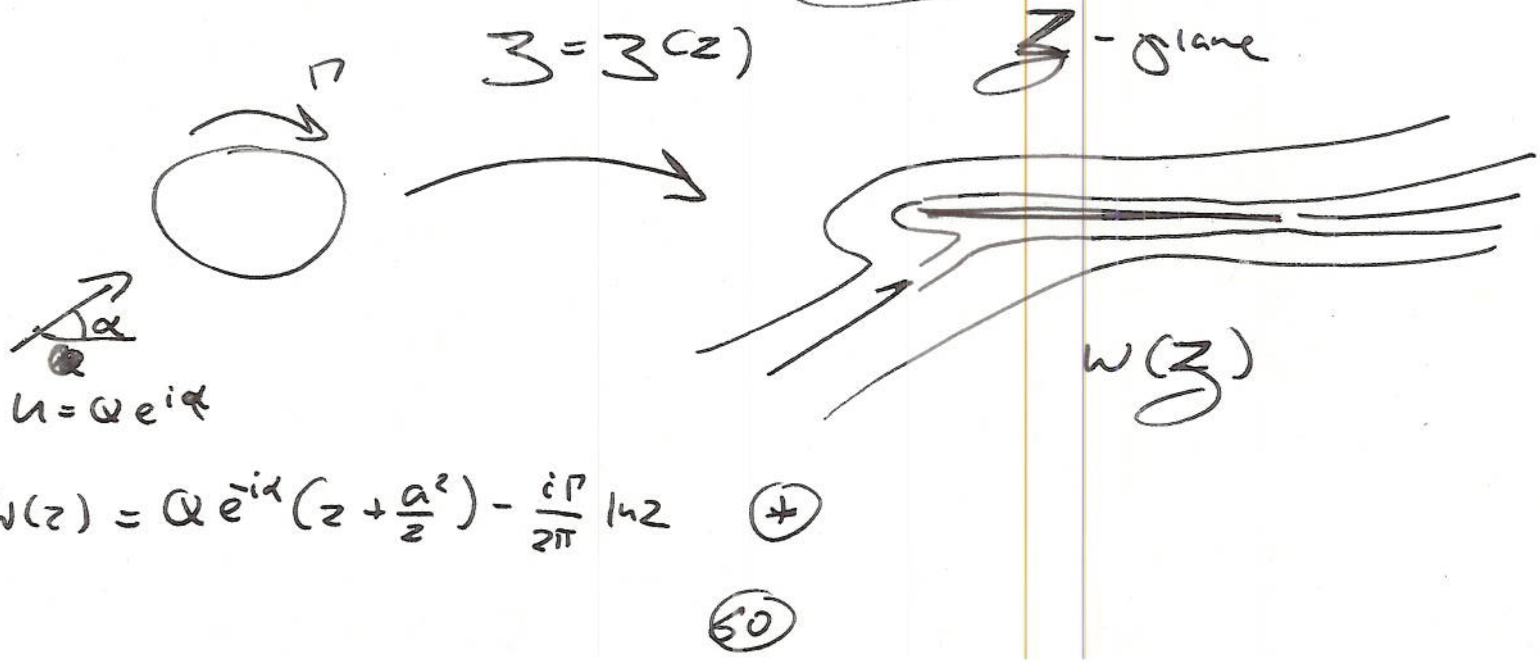
We have a "paradox", because this result clearly contradicts the behavior observed in reality, revealing unphysical nature of ideal flow models.

* the lift is $F_y = -\rho \Gamma U$
and is proportional to the circulation, which is what one observes in reality.

Q: How do determine the circulation Γ?
* for a circular cylinder it can be, in principle, arbitrary; one could fix it by prescribing the separation angle



* consider flow past a flat plate as a model of an airfoil



$$w(z) = Q e^{-i\alpha}\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z \quad (*)$$

Flow in the physical (ζ) plane

$$V(\zeta) = \frac{dW(\zeta)}{d\zeta} = \frac{dw}{dz} \Big/ \frac{d\zeta}{dz}$$

where:

- $\frac{dw}{dz}$ is given by (*)
- $\frac{d\zeta}{dz}$ is obtained from the conformal map $\zeta(z) = z + \frac{a^2}{z}$ transforming the exterior of the circle into the exterior of a finite plate of length 4a

$$V(z) = \frac{Ue^{-i\alpha}\left(1 - \frac{a^2}{z^2}\right) - \frac{i\Gamma}{2\pi z}}{\left(1 - \frac{a^2}{z^2}\right)} \quad (**)$$

Note that, for arbitrary Γ, $|V(z)| \rightarrow \infty$ as $z \rightarrow \pm a$ or $\zeta \rightarrow \pm 2a$

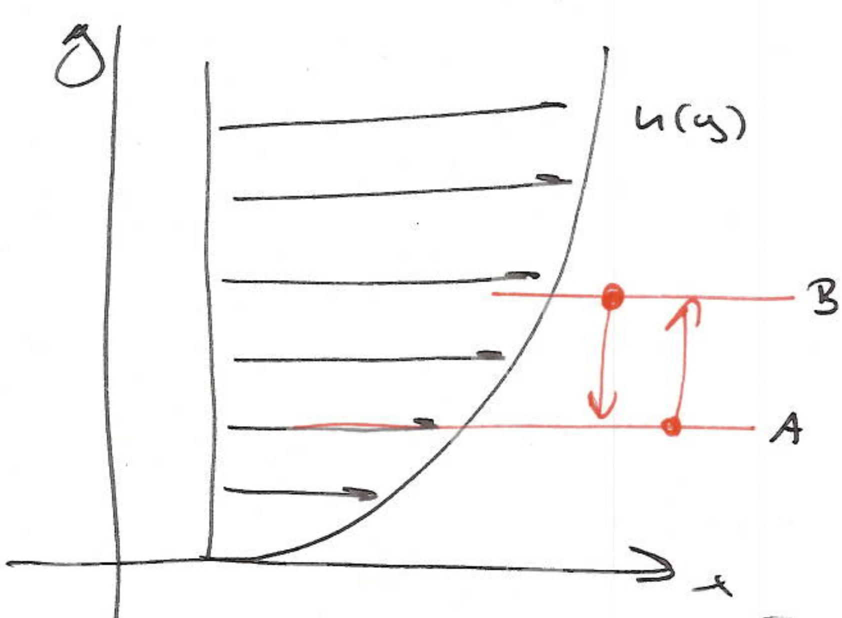
Thus, the complex velocity is unbounded at the endpoints of the plate. One of these singularities can be removed by fixing the circulation Γ, so that the numerator in (**) vanishes for $z = a$ (trailing edge). This is equivalent to prescribing the "separation point" (i.e., the position where the streamlines separate from the body) $\Rightarrow$ Kutta / Kutta-Joukowski condition

In general, the Kutta–Joukowsky condition allows one to determine the circulation in a 2D steady potential flow, so that the separation occurs where expected in a real flow (i.e., trailing edge). This, in turn makes it possible to compute lift. Fig.

Remark

* A more complicated conformal map, the so-called Joukowsky transformation, can transform the potential flow past a cylinder to a flow past an airfoil

3) VISCOSITY & NAVIER-STOKES EQUATION



Consider the flow field $[u(y), 0]$

In real fluids when a fluid particle moves between levels A and B, it must be accelerated or decelerated. Since ~~[illegible]~~ fluid momentum is changed, there must be a force → it acts to reduce the differences in the velocity profile. Empirically, for many fluids it can be represented as

$$(*) \qquad F(y) = \mu \frac{du(y)}{dy}$$

where μ – viscosity (material property)

Remark

Note that ~~the~~ constitutive relation $(*)$ does not depend on the velocity, but on its gradient (as a measure of deformation). Fluids which behave as described by $(*)$ are called "Newtonian".

Rheology studies ~~[illegible]~~ the constitutive relations which are not Newtonian.

In more general flows we have the following stress tensor

$$\sigma_{ij} = \underbrace{-p\delta_{ij}}_{\substack{\text{already present} \\ \text{in Euler equations}}} + \underbrace{\Pi_{ij}}_{\substack{\text{viscous} \\ \text{stress tensor}}}, \quad i,j = 1,2,3$$

It can be shown that in ~~micropolar~~ fluids without micropolar effects the viscous stress tensor must be symmetric

$$\Pi_{ij} = \Pi_{ji}$$

Constitutive relation for Newtonian fluid

$$(17) \quad \Pi_{ij} = \underbrace{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)}_{\Pi^0_{ij}} + \underbrace{\mu' \frac{\partial u_k}{\partial x_k}\delta_{ij}}_{\Pi'_{ij}} \qquad i,j = 1,2,3$$

Remarks

* the first part Π^0_{ij} has the most general form which is

- symmetric $\Pi^0_{ij} = \Pi^0_{ji}$
- linear
- deviatoric $Tr(\Pi^0) = 0$ (it implies that ~~...~~ $\bar{\bar{\Pi}}^0$ does not contribute to the normal force on an area element $\bar{n}\bar{\bar{\Pi}}^0\bar{n} = 0$)

* Since $\frac{\partial u_k}{\partial x_k} = \nabla\cdot\bar{u}$, (17) can be rewritten as

$$\bar{\bar{\Pi}} = \mu\bar{\bar{\Pi}}^0 + \mu'(\nabla\cdot\bar{u})I$$

μ' - second viscosity (characterizes dissipation due to ~~...~~ pure compression)