For many fluids it is assumed that $\mu' = 0$
(Stokes hypothesis)

* in the <u>incompressible</u> case ($\nabla \cdot \bar{u} = 0$)

$$\Pi_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

OR $\quad \bar{\bar{\Pi}} = 2\mu \bar{\bar{D}} \quad , \quad \underbrace{\bar{\bar{D}} = \frac{1}{2}\left[\nabla \bar{u} + (\nabla \bar{u})^T \right]}_{\text{rate of strain tensor}}$

<u>Digression</u>

Newtonian fluids vs. Hookean solids – in linearly elastic materials stresses depend on the deformation, rather than deformation velocity (rate of strain) as in fluids

Navier-Stokes Equation

Consider the general form of our equation of momentum conservation

$$\rho \frac{d\bar{u}}{dt} = \rho \left(\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla)\bar{u} \right) = \nabla \cdot \bar{\bar{\sigma}}$$

$$\nabla \cdot \bar{u} = 0$$

The new constitutive hypothesis

$$\bar{\bar{\sigma}} = -pI + \bar{\bar{\Pi}} = -pI + 2\mu \bar{\bar{D}}$$

$$\nabla \cdot \bar{\bar{\sigma}} = -\nabla p + 2\tfrac{1}{2}\mu\left(\nabla\cdot\nabla\bar{u} + \nabla\cdot(\nabla\bar{u})^T\right)$$

$$= -\nabla p + \mu \Delta \bar{u}$$

$$\frac{\partial}{\partial x_j}\frac{\partial u_j}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} = 0$$

$$\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j} = \Delta u_i$$

Finally, the Navier-Stokes system

$$(18)\quad \begin{cases} \rho\left(\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u}\right) = -\nabla p + \mu\Delta\bar{u} \\ \nabla\cdot\bar{u} = 0 \end{cases}$$

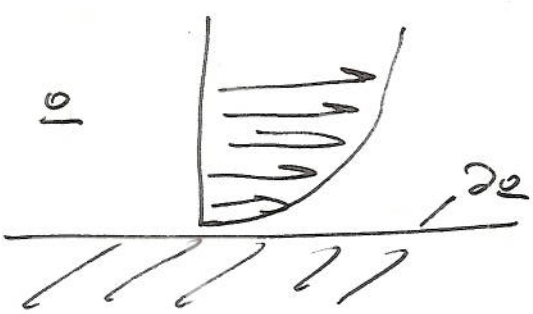
Dividing into ρ, we obtain another form

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u} = -\frac{1}{\rho}\nabla p + \nu\Delta\bar{u}$$

$\nu = \frac{\mu}{\rho}$ - kinematic viscosity

* Boundary conditions for Navier - Stokes system

Solid boundary

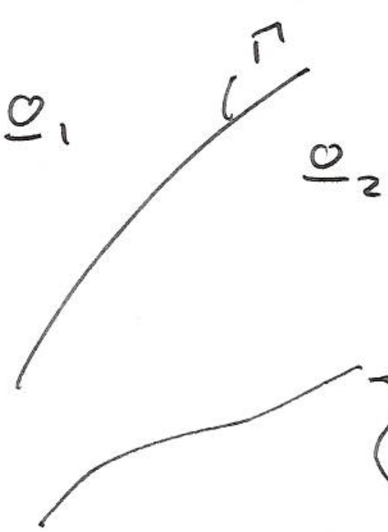


Now we have a second-order operator Δ applied to a vector field

↳ need a separate BC for each scalar field

$\bar{V}|_{\partial\Omega} = 0$ (no-slip)

BCs on the interface between two immiscible NS fluids



We need three scalar conditions on each side:

$$\left.\bar{V}\right|_1 = \left.\bar{V}\right|_2 \quad \text{on } \Gamma$$

$$\left.\bar{\bar{\sigma}}\cdot\bar{n}\right|_1 = \left.\bar{\bar{\sigma}}\cdot\bar{n}\right|_2$$

conservation of mass & momentum

If surface tension (characterized by coefficient γ is present)

$$\left.\bar{\bar{\sigma}}\cdot\bar{n}\right|_1 - \left.\bar{\bar{\sigma}}\cdot\bar{n}\right|_2 = \gamma\, \varkappa\, \bar{n}$$

$\varkappa$ - curvature

Dynamic similarity

Suppose the characteristic dimension is L and the characteristic velocity is U

Consider non-dimensionalized variables

$$\bar{u}^* = \frac{\bar{u}}{U}, \quad \bar{x}^* = \frac{\bar{x}}{L}, \quad p^* = \frac{p}{gu^2}, \quad t^* = \frac{t}{L/U}$$

and define the Reynolds number $Re = \frac{UL}{\nu}$

then, in terms of these new variables the NS system becomes

$$\begin{cases} \dfrac{\partial \bar{u}^*}{\partial t^*} + (\bar{u}^* \cdot \nabla^*)\bar{u}^* = -\nabla^* p^* + \dfrac{1}{Re}\Delta^* \bar{u}^* \\ \nabla^* \cdot \bar{u}^* = 0 \end{cases}$$

The Reynolds number is the only parameter present in the nondimensionalized system

Conclusion (dynamical self-similarity)

Two flows in

- geometrically similar domains Ω_1 and Ω_2 and
- with the same Reynolds numbers $Re_1 = Re_2$

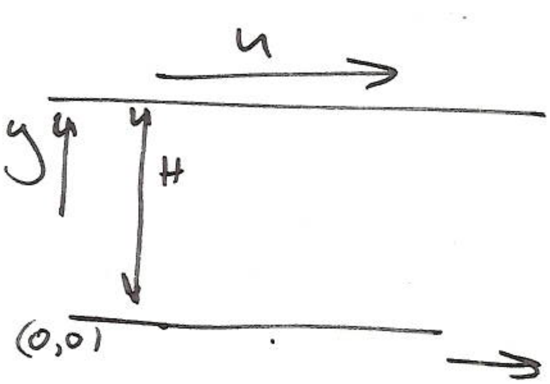
are equivalent

$$Re \sim \frac{|\rho(\bar{u}\cdot\nabla)\bar{u}|}{|\mu\Delta\bar{u}|} = \frac{[\text{inertial (nonlinear) forces}]}{[\text{viscous forces}]}$$

* inviscid limit : $Re \to \infty$ — formally, Euler equation is recovered, but for any finite Re the no-slip boundary conditions still apply (singular perturbation problem)

* Navier-Stokes system — some analytical solutions

Couette Flow



Assume

$\rho = \text{const}$

$\bar{u} = [u(y), 0]$

$\frac{\partial p}{\partial x} = 0$

Since $v \equiv 0$, the Y-momentum eq. is satisfied trivially.

The continuity equation: $\frac{\partial u(y)}{\partial x} + \frac{\partial v}{\partial y} = 0$

also trivially

X-component of momentum equation

$$\underbrace{\frac{\partial u}{\partial t}}_{0} + \underbrace{u\frac{\partial u}{\partial x}}_{0} + \underbrace{v\frac{\partial u}{\partial y}}_{0} = \underbrace{-\frac{\partial p}{\partial x}}_{0} + \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u$$

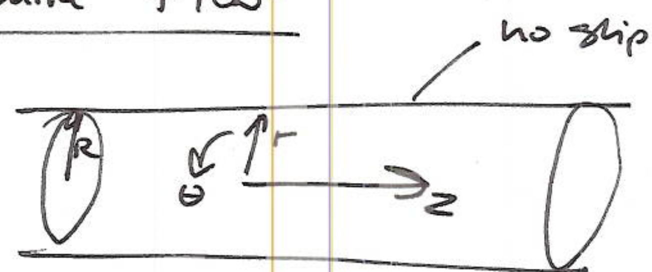
Thus, we have $\nu \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u(y) = Ay + B$

Matching with the BCs $u(0) = 0$, $u(H) = U$

$$u(y) = \frac{U}{H} y \quad \text{(linear profile)}$$

Poiseuille / Hagen-Poiseuille Flow

Consider the flow in cylindrical pipe with radius R



Use cylindrical coordinate system with $\bar{u} = [u_z, u_r, u_\theta] = [u_z(r), 0, 0]$

Assume

Need to rewrite Navier-Stokes equations in the cylindrical geometry

Some auxilliary notation:

- cylindrical Laplacian $f = f(z, r, \theta)$

$$\Delta f = \frac{\partial^2 f}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

- ~~conv~~ cylindrical advection operator

$$\bar{u}\cdot\nabla = u_z\frac{\partial}{\partial z} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}$$

define also $\delta = \Delta - \frac{1}{r^2}$

Then, the momentum and continuity equations become

$$\frac{\partial u_z}{\partial t} + (\bar{u}\cdot\nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Delta u_z$$

$$\frac{\partial u_r}{\partial t} + (\bar{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\delta u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$\frac{\partial u_\theta}{\partial t} + (\bar{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\delta u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right)$$

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0$$

Since $u_r = u_\theta = 0$ and $u_z = u_z(r)$, the r and θ ~~momentum~~ component of the momentum equation and the continuity equation are satisfied trivially.

We are left with the z-component momentum eq. which becomes

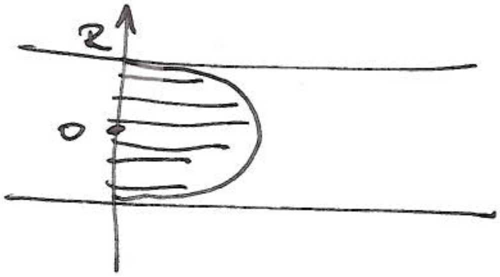
$$\nu\Delta u_z = \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Assume $p(z) = -Gz + C_0$

where G is the pressure drop over a unit of length

Then $\mu \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} \right) = -G$

single linear 2nd-order ODE for $u_z = u_z(r)$



Boundary conditions

$\frac{\partial u_z}{\partial r} = 0 \qquad r = 0$

$u_z = 0 \qquad r = R$

Solution: $u_z(r) = \frac{G}{4\mu}(R^2 - r^2)$, $r \in [0, R]$

(parabolic profile)

Total flux: $Q = \int_A u_z \, d\sigma = 2\pi \int_0^R r u_z \, dr = \frac{\pi G R^4}{8\mu}$

depends on:

- pressure drop G
- pipe radius R
- viscosity μ

Remark

The parabolic flow is not a unique solution of this problem, but is the only one known in analytic form. When the Reynolds number $Re = \frac{u_{max} \cdot (2R)}{(\mu/\rho)}$ is sufficiently large

($Re \cong 2300$)

it becomes unstable and transitions to a turbulent flow. Reynold's 1870s pipe flow experiment was the first scientific observation of the transition to turbulence.

* Viscous vorticity equation

$$\frac{\partial \bar{\omega}}{\partial t} + (\bar{u}\cdot\nabla)\bar{\omega} = \bar{\omega}\cdot\nabla\bar{u} + \nu\Delta\bar{\omega}$$

Many properties of the inviscid vorticity equation still apply.