it becomes unstable and transitions to a turbulent flow. Reynold's 1870s pipe flow experiment was the first scientific observation of the transition to turbulence.

* Viscous vorticity equation

$$\frac{\partial \bar{\omega}}{\partial t} + (\bar{u} \cdot \nabla) \bar{\omega} = \bar{\omega} \cdot \nabla \bar{u} + \nu \Delta \bar{\omega}$$

Many properties of the inviscid vorticity equation still apply. In particular, boundary conditions are also given in terms of velocity.

* Some special vortex flows

Consider velocity field given in the cylindrical polar coordinate system by

$$\bar{u} = [u_z, u_r, u_\theta] = [0, 0, u_\theta(r,t)]$$

The momentum equation reduces to

$$\frac{\partial p}{\partial r} = \frac{\rho}{r} u_\theta^2$$

The vorticity is $\omega_z = \frac{1}{r} \frac{\partial}{\partial r}(r u_\theta)$, $\bar{\omega} = [\omega_z(r), 0, 0]$

and the vorticity equation becomes

$$\frac{\partial \omega}{\partial t} = \nu \left(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} \right) = \nu \Delta \omega$$

Thus, in 2D flows with ~~axsim~~ azimuthal symmetry vorticity diffuses in the same way as heat.

Suppose the diffusing vortex is embedded in a stretching field $\bar{u}_0 = \left[\alpha z, -\frac{\alpha r}{2}, 0\right]$, so that the total field is $(\alpha \in \mathbb{R})$

$$\bar{u} + \bar{u}_0 = \left[\alpha z, -\frac{\alpha r}{2}, u_\theta(r,t)\right]$$

The vorticity equation it satisfies takes the form

$$\frac{\partial \omega}{\partial t} \underbrace{- \frac{\alpha r}{2}\frac{\partial \omega}{\partial r} - \alpha\omega}_{\text{advection}} = \underbrace{\nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right)}_{\text{diffusion}}$$

Assume steady state: $\frac{\partial \omega}{\partial t} = 0$

Thus

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\alpha}{2}r^2\omega + \nu r \frac{\partial \omega}{\partial r}\right) = 0$$

Integrate & assume $\left.\begin{matrix} r^2\omega \\ r\frac{\partial \omega}{\partial r}\end{matrix}\right\} \xrightarrow[r\to\infty]{} 0$

$$\frac{\alpha}{2} r\omega + \nu \frac{d\omega}{dr} = 0 \Rightarrow \underbrace{\omega(r) = C e^{-\frac{\alpha r^2}{4\nu}}}_{}$$

$\omega(r)$ approaches a Dirac delta function (a point vortex) when $\nu \to 0$ or $\alpha \to \infty$

equilibrium vorticity profile representing the balance between ~~stre~~ advection and diffusion

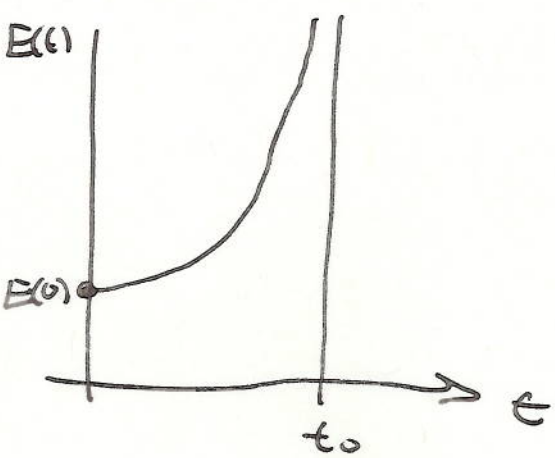
What wins in real flows : nonlinear stretching or viscous dissipation?

We don't know ...

Consider enstrophy $E(t) = \int_{\Omega} |\bar{\omega}(t,\bar{x})|^2 d\underline{\Omega}$

How much can it grow in a viscous flow?
The best result available to-date is

$$\frac{dE(t)}{dt} \leq C E(t)^3$$



It allows for finite-time blow-up (singularity formation), but we do not know if this estimate is sharp.

* Turbulence (an informal introduction)

Consider spectral decomposition (Fourier transform) of the velocity field

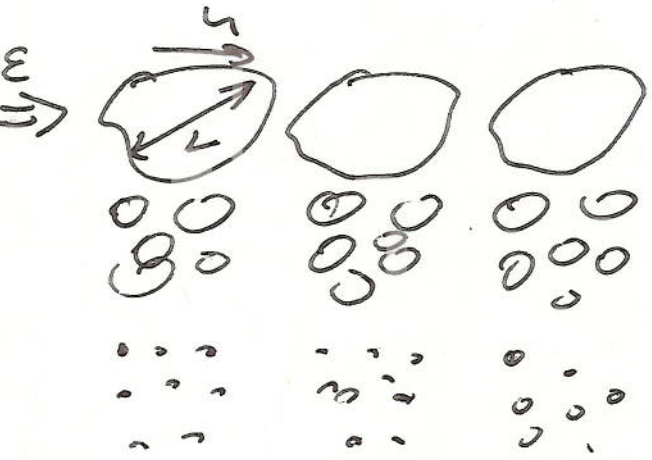
$$\bar{u}(t,\bar{x}) = \sum_{\bar{k} \in \mathbb{Z}^3} \hat{u}_{\bar{k}} \, e^{i k \cdot x}$$

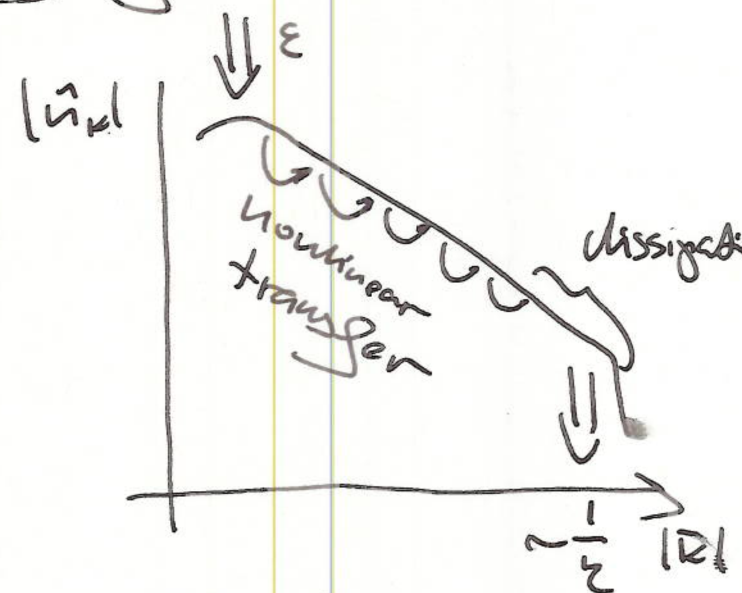
$\hat{u}_{\bar{k}} \in \mathbb{C}$ - Fourier coefficients

U, L - velocity and length scales ~~assoc~~ associated with "stirring"

Assume the flow is defined on an ~~box~~ unbounded domain, is isotropic and homogeneous.

ε - energy of stirring per unit volume





Vortex stretching transfers energy to smaller scales of motion.

How small can they become?

To estimate the Kolmogorov length scale, use dimensional analysis (not rigorous!)

Kolmogorov length scale η is the eddy size for which the Reynolds number is close to unity (so that viscous forces start to dominate)

$$Re_\eta = \frac{u_\eta \eta}{\nu} \simeq 1$$

, u_η - corresponding velocity

From dimensional analysis

$$\varepsilon \sim \frac{u^3}{L} \quad (*)$$

On the other hand

$$\left.\begin{aligned} \varepsilon &\sim \nu \frac{u_\eta^2}{\eta^2} \\ u_\eta &\sim \frac{\nu}{\eta} \end{aligned}\right\} \Rightarrow \varepsilon \sim \frac{\nu}{\eta^2}\left(\frac{\nu}{\eta}\right)^2 = \frac{\nu^3}{\eta^4} \Rightarrow$$

$$\Rightarrow \eta = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}$$

Combining this with $(*)$, we obtain

$$\frac{\eta}{L} \sim Re^{-3/4}, \quad \text{where } Re = \frac{uL}{\nu}$$

This is a 1D result.

Although purely phenomenological, this estimate is important as it gives an idea about the size of the smallest structures in a turbulent flow with a given Reynolds number. This determines the numerical accuracy required to resolve such flows: $\Delta x \sim \eta$

Number of grid points (in 1D): $N \sim \frac{L}{\Delta x} \sim Re^{3/4}$

In 3D: $N_{3D} \sim N^3 \sim Re^{9/4}$

This is an "order of magnitude" analysis.

3b) STOKES FLOWS

Many important applications concern the case when:

- the length scales are extremely ~~small~~ small, or
- the velocities are very small, or
- the viscosities are very large.

In any of those cases $Re = \frac{UL}{\nu} << 1$

We will derive and analyze the corresponding

Stokesian approximation

Consider the Navier-Stokes system in the non-dimensional form

$$Re\left[\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u} + \nabla p\right] = \Delta \bar{u}$$

$$\nabla\cdot\bar{u} = 0$$

Then, letting $Re \to 0$

$$\begin{cases} \Delta \bar{u} = 0 \\ \nabla\cdot\bar{u} = 0 \end{cases}$$

The system appears over-determined (4 scalar equations vs. 3 variables)

In the above formulation, pressure was nondimensionalized as

$$p^* = \frac{P}{\rho u^2}$$

If we do this differently, namely

$$p^* = \frac{L}{\mu U} p \text{, then}$$

$$\begin{cases} Re\left[\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u}\right] + \nabla p = \Delta \bar{u} \\ \nabla\cdot\bar{u} = 0 \end{cases}$$

and

(19) $\begin{cases} \nabla p - \Delta\bar{u} = 0 \\ \nabla\cdot\bar{u} = 0 \\ \text{no slip BCs} \end{cases}$ Stokes system

What can be said about is solutions?

Let $\bar{u} = \nabla\phi + \bar{v}$, where

$$\Delta\bar{v} = 0$$

then

$$\nabla p - \Delta(\nabla\phi + \bar{v}) = 0 \Rightarrow \nabla\Delta\phi = \nabla p$$

$$\left.\begin{array}{l}\Delta\phi = p \\ \Delta\phi = -\nabla\cdot\bar{v}\end{array}\right\} \nabla\cdot\bar{v} = -p$$

Stokes problem (19) is linear and we can solve it (in principle)

$$\Delta p = \nabla\cdot\nabla p = \Delta\nabla\cdot\bar{u} = 0$$

$$\Delta(\nabla p - \Delta\bar{u}) = 0 \Rightarrow \Delta^2\bar{u} = 0 \text{ – biharmonic equation}$$

Since $\Delta\bar{v} = 0$, we can rewrite this as $\Delta^2\phi = 0$

We can construct the fundamental solution to

$$\begin{cases} \nabla p - \Delta \bar{u} = \bar{F}\,\delta(\bar{x}) \\ \nabla\cdot\bar{u} = 0 \end{cases}, \quad \bar{F}\in\mathbb{R}^3 \text{ some vector}$$

equivalently $\Delta^2 \chi = \delta(\bar{x})$

which has the fundamental solution

$$\chi(\bar{x},\bar{y}) = \frac{|\bar{x}-\bar{y}|}{8\pi} + A + \frac{B}{|\bar{x}-\bar{y}|}, \quad A, B \text{ - constants}$$

Transforming back to the velocity-pressure representation:

$$\left.\begin{aligned} u_i &= \frac{1}{8\pi}\left[\frac{(x_i-y_i)(x_j-y_j)}{|\bar{x}-\bar{y}|^3} + \frac{\delta_{ij}}{|\bar{x}-\bar{y}|}\right]F_j \\ p &= \frac{1}{4\pi}\,\frac{(x_j-y_j)F_j}{|\bar{x}-\bar{y}|^3} \end{aligned}\right\} \text{ Stokeslet}$$

Remarks

* this approach can be used to compute the Stokes flow past a sphere (of radius R) in 3D leading to the famous drag formula

$$D = 6\pi\mu a U, \quad U \text{ - oncoming flow at infinity}$$

* Stokes Paradox – in 2D the Stokes flow
  - satisfying the no-slip BCs on the obstacle, and
  - approaching the uniform free stream at infinity

does not exist.

The reason is that in 2D the assumptions underlying Stokes approximation are not uniformly valid everywhere. (The region of fluid slowed down by the obstacle extends to infinity)

The situation is remedied by the Oseen approximation

$$U\frac{\partial \bar{u}}{\partial x} = -\nabla p + \Delta \bar{u}$$
$$\nabla \cdot \bar{u} = 0$$

in which a linear advection term is restored. This system can be solved in terms of "Oseenlets"