* Stokes Paradox - in 2D the Stokes flow
  - satisfying the no-slip BCs on the obstacle, and
  - approaching the uniform free stream at infinity

  does not exist.

The reason is that in 2D the assumptions underlying Stokes approximation are not uniformly valid everywhere. (The region of fluid slowed down by the obstacle extends to infinity)

The situation is remedied by the Oseen approximation

$$U\frac{\partial \bar{u}}{\partial x} = -\nabla p + \Delta \bar{u}$$

$$\nabla \cdot \bar{u} = 0$$

in which a linear advection term is restored. This system can be solved in terms of "Oseenlets"

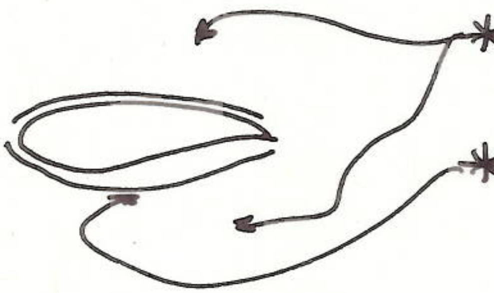
(3c) BOUNDARY LAYERS

~~[illegible]~~ We have seen that purely potential flows do not account for many real flows.

Question - how to account for the effects of viscosity in high-Re limit?

Proposal made by Ludwig Prandtl in 1904

* A flow past an obstacle at a high Reynolds number can be split into two parts:



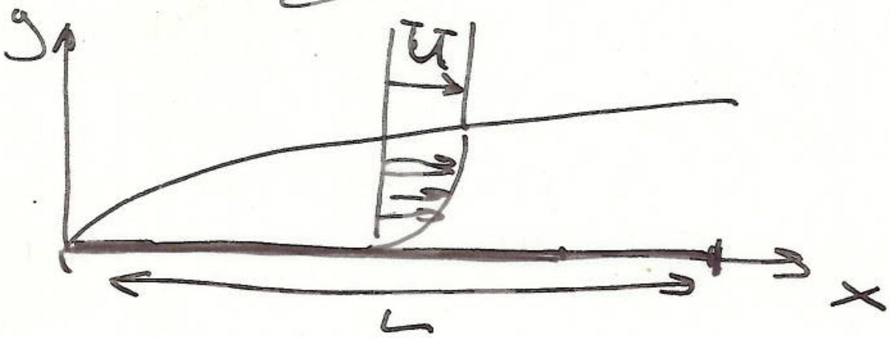
* potential exterior flow
* thin viscous "boundary layer" adjacent to the obstacle

This theory is meant to be valid in the limit $Re \to \infty$.

How do we describe the boundary layer mathematically?

We will obtain a simplification of the Navier-Stokes equations valid close to the obstacle

Consider steady, incompressible 2D flow past a ~~semi-infinite~~ finite plate



$$(\bar{u}\cdot\nabla)u + \frac{\partial p}{\partial x} - \frac{1}{Re}\Delta u = 0$$

$$(\bar{u}\cdot\nabla)v + \frac{\partial p}{\partial y} - \frac{1}{Re}\Delta v = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Make assumptions and estimate the order of magnitude of the different terms:

* $(\bar{u}\cdot\nabla)u \sim \frac{U^2}{L} \sim O(1)$ for large Re
* $\frac{\partial^2 u}{\partial y^2} \gg \underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{can be neglected}}$

* The thickness of the boundary layer vanishes as $Re \to \infty$; we thus rescale the vertical coordinate and velocity component as

$$\tilde{y} = \sqrt{Re}\, y \quad , \quad \tilde{v} = \sqrt{Re}\, v$$

The continuity equation thus becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \qquad (20a)$$

The x-component of the momentum equation ($\frac{1}{Re}\frac{\partial^2 u}{\partial x^2}$ dropped)

$$u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} + \frac{\partial p}{\partial x} - \frac{\partial^2 u}{\partial \tilde{y}^2} = 0 \qquad (20b)$$

(independent of Re!)

The y-component of the momentum equation

$$\frac{\partial p}{\partial \tilde{y}} = -\frac{1}{Re}\left[u\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} - \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right] + \frac{1}{Re^2}\frac{\partial^2 \tilde{v}}{\partial x^2}$$

After taking the limit $Re \to \infty$

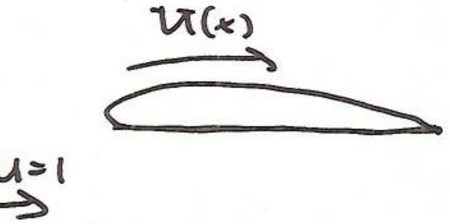
$$\frac{\partial p}{\partial \tilde{y}} = 0 \implies p = F(x) \qquad (20c)$$

Equations (20a) – (20c) are <u>referred</u> to as Prandt equations of the boundary layer

<u>Remarks</u>:

* pressure in the boundary layer is constant

and equal to the pressure just outside the layer (i.e., in the potential flow)



from Bernoulli

$p_1 = p_0 + \frac{1}{2} - \frac{1}{2} U^2(x)$

flow past a flat plate

$U(x) = 1 \Rightarrow p_1 = \text{const.}$

* ~~[illegible]~~ equation (20b) is reminiscent of a parabolic (heat) equation with the x coordinate playing the role of "time" $\Rightarrow$ the boundary layer thickens as one moves downstream

* system (20a)-(20c) is nonlinear and in some circumstances solutions may cease to exist $\Rightarrow$ <u>separation</u> (attached flow can no longer be computed)

* the above property suggest the following approach to solving the flow problem:

- ~~[illegible]~~ compute potential flow & find pressure
- solve boundary layer equations and compute drag, etc.
- if separation is detected, recompute the potential flow to account for it

$\left.\right\}$ iterate until convergence

~~[illegible]~~