# QUIZ #1

10:30am, October 25 (Friday), 20 minutes, 10 points max
(no books, no notes)

Write your name and Email address on the top of this sheet
Write your answers on the reverse side and/or attach additional sheets as necessary.

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1. Suppose the flow domain is two-dimensional (2D) and unbounded , i.e., $\Omega = \mathbb{R}^2$. Given a suitable vorticity field $\omega = \omega(x, y)$ and knowing that the fundamental solution of the Laplace equation in 2D has the form

$$G(x, y, x', y') = -\frac{1}{2\pi} \ln \sqrt{(x-x')^2 + (y-y')^2},$$

derive the corresponding Biot-Savart law allowing one to determine the velocity field $\mathbf{u} = [u, v]$, such that $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.

Consider the streamfunction $\psi : \underline{\Omega} \to \mathbb{R}$.

Then $\bar{u} = \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right]$

The streamfunction and vorticity are related via Poisson eq.

$$-\Delta \psi = \omega \quad \text{in } \underline{\Omega}$$

so that $\psi(\bar{x}) = -\Delta^{-1}\omega = -\int_{\underline{\Omega}} G(\bar{x}, \bar{x}')\,\omega(\bar{x}')\,d\underline{\Omega}(x')$

$$= \frac{1}{2\pi}\int_{\underline{\Omega}} \ln\sqrt{(x-x')^2+(y-y')^2}\,\omega(x',y')\,dx'dy$$

Since $\underline{\Omega} = \mathbb{R}^2$, we don't have to explicitly account for the boundary conditions

The Biot-Savart law is obtained by computing the partial derivatives

$$\bar{u}(\bar{x}) = \frac{1}{2\pi}\left[\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right]\int_{\underline{\Omega}} \ln\sqrt{(x-x')^2+(y-y')^2}\,\omega(x',y')\,dx'dy'$$

$$= \boxed{\frac{1}{2\pi}\int_{\underline{\Omega}} \frac{[(y-y'), -(x-x')]}{r^2}\,\omega(x',y')\,dx'dy'}, \text{ where } r^2 = (x-x')^2 + (y-y')^2$$