EHRENFEUCHT-FRAÏSSÉ GAMES IN CONTINUOUS LOGIC

BRADD HART

For a finite set of L-formulas F in the variables $x_1, \ldots, x_n$ and $\epsilon > 0$, $EF(M, N, \epsilon, F, n)$ is an n-step game played between two L-structures M and N. At stage i of the game, Player I picks either $a_i \in M$ or $b_i \in N$ and Player II responds by playing $b_i \in N$ or respectively $a_i \in M$. After n stages, two sequences will have been produced $a_1, \ldots, a_n$ in M and $b_1, \ldots, b_n$ in N. Player II wins the game if for every $\varphi \in F$, $|\varphi^M(a_1, \ldots, a_n) - \varphi^N(b_1, \ldots, b_n)| < \epsilon$.

Theorem 1. *The following are equivalent for two* L*-structures* M *and* N*:*

(1) $M \equiv N$.
(2) *For all* $\epsilon > 0$, n *and finite set of formulas* F, *Player II has a winning strategy for* $EF(M, N, \epsilon, F, n)$.
(3) *For all* $\epsilon > 0$, n *and finite set of atomic formulas* F, *Player II has a winning strategy for* $EF(M, N, \epsilon, F, n)$.

Proof. Condition (2) implies (3) clearly and (2) implies (1) follows by letting $n = 0$ and F be any L-sentence. Toward a proof of (1) implies (2) we adopt the following notation.

Notation For $\epsilon > 0$, F a set of L-sentences and two L-structures M and N define $M \equiv_\epsilon^F N$ iff $|\varphi^M - \varphi^N| < \epsilon$ for all $\varphi \in F$. We will write L_c for the language L together with a new constant symbol c. $L_{c_1,\ldots,c_n}$ is L with n new constant symbols.

Lemma 2. *Suppose that* $F = \{\varphi_1(c), \ldots, \varphi_k(c)\}$ *for a language* L_c *and* $\epsilon > 0$. *Then there is a finite set* $\tilde{F}$ *of* L*-sentences so that if* $M \equiv_\epsilon^{\tilde{F}} N$ *then for every* $a \in M$ *there is a* $b \in N$ *such that* $(M, a) \equiv_{3\epsilon}^F (N, b)$.

Proof. Fix an ϵ-dense set $r_1^i, \ldots, r_\ell^i$ in the range of $\varphi_i(x)$. Suppose that $S : \{1, \ldots, k\} \to \{1, \ldots, \ell\}$ and define $\theta_S(x)$ to be

$$\max_i(|\varphi_i(x) - r_{S(i)}^i| \dot{-} \epsilon).$$

Let $\tilde{F}$ be the set of L-sentences $\inf_x \theta_S(x)$ as S ranges over all possible functions.

Now suppose that $M \equiv_\epsilon^{\tilde{F}} N$ and $a \in M$. Choose S so that

$$|\varphi_i^M(a) - r_{S(i)}^i| \le \epsilon$$

for all i. By the $(\epsilon, \tilde{F})$-equivalence, there is $b \in N$ so that $\theta_S^N(b) \leq \epsilon$. So for each i we have

$$|\varphi_i^M(a) - r^i_{S(i)}| \leq \epsilon \text{ and } |\varphi_i^N(b) - r^i_{S(i)}| \leq 2\epsilon$$

so

$$|\varphi_i^M(a) - \varphi_i^N(b)| \leq 3\epsilon.$$

□

Now we proceed to give a winning strategy for Player II in the game $EF(M, N, \epsilon, F, n)$ where $M \equiv N$ and

$$F = \{\varphi_1(x_1, \ldots, x_n), \ldots, \varphi_k(x_1, \ldots, x_n)\}.$$

First we define a sequence of sets of sentences F_i for $i = 0, \ldots, n$.

$$F_n = \{\varphi_1(c_1, \ldots, c_n), \ldots, \varphi_k(c_1, \ldots, c_n)\}$$

is a set of sentence in $L_{c_1,\ldots,c_n}$. If we have defined F_{i+1} as a finite set of sentences in $L_{c_1,\ldots,c_{i+1}}$ then let F_i be $\tilde{F}_{i+1}$ from the previous lemma.

At each stage of the strategy we will guarantee that if $a_1, \ldots, a_i \in M$ and $b_1, \ldots, b_i \in N$ have been picked then

$$(*) \qquad (M, a_1, \ldots, a_i) \equiv^{F_i}_{\epsilon/3^{n-i}} (N, b_1, \ldots, b_i).$$

We include the case of $i = 0$: Since $M \equiv N$ then we definitely have $M \equiv^{F_0}_{\epsilon/3^n} N$. Now assume that $(*)$ holds and Player I has picked a_{i+1} from M. Then by the lemma and the definition of F_{i+1} we can choose b_{i+1} so that

$$(M, a_1, \ldots, a_{i+1}) \equiv^{F_{i+1}}_{\epsilon/3^{n-i+1}} (N, b_1, \ldots, b_{i+1}).$$

The case where Player I chooses from N is symmetric. So in the end we have

$$(M, a_1, \ldots, a_n) \equiv^{F_n}_{\epsilon} (N, b_1, \ldots, b_n)$$

and so Player II wins the game. This finishes the proof of (1) implies (2).

We finish the proof of the theorem by showing that (3) implies (2). We do this by induction on quantifier depth.

Definition 3. *We define the quantifier depth* $qd(\varphi)$ *of a formula* φ *by induction on formulas:*

(1) *Atomic formulas have quantifier depth 0.*
(2) *If* $\varphi = f(\psi_1, \ldots, \psi_k)$ *for formulas* $\psi_1, \ldots, \psi_k$ *and* f *is a continuous function then* $qd(\varphi) = \max_i qd(\psi_i)$.
(3) *If* $\varphi = \sup_x \psi$ *or* $\inf_x \psi$ *then* $qd(\varphi) = qd(\psi) + 1$.

Lemma 4. *Every* L*-formula is equivalent to one of the form* $f(\psi_1, \ldots, \psi_k)$ *where each* ψ_i *is either an atomic formula or a formula of the form* $\inf_x \theta$.

Proof. By induction on formulas together with the fact that $\sup_x \theta$ is logically equivalent to $-\inf_x(-\theta)$. □

Now let (2′) be the statement that for all $\epsilon > 0$, n and finite set of formulas F containing formulas of the form $\inf_x \theta$ or atomic formulas, Player II has a winning strategy for $\mathrm{EF}(M, N, \epsilon, F, n)$. We show that (2′) implies (2). For suppose that F, ϵ and n are given and that

$$F = \{f_1(\psi^1_1, \ldots, \psi^1_{k_1}), \ldots, f_m(\psi^m_1, \ldots, \psi^m_{k_m})\}$$

where $f_1, \ldots, f_m$ are continuous functions and ψ^i_j is either of the form $\inf_x \theta$ or is atomic. If we let $\delta > 0$ be the minimum of the uniform continuity moduli when the f_i's are restricted to the ranges of the ψ^i_j's, we see that the winning strategy for Player II in the game $\mathrm{EF}(M, N, \delta, F', n)$ where

$$F' = \{\psi^1_1, \ldots, \psi^m_{k_m}\}$$

is also a winning strategy for Player II in $\mathrm{EF}(M, N, \epsilon, F, n)$. We now proceed to prove (2′) by induction on the quantifier depth of F which is the maximum of the quantifier depth of the formulas in F.

The base case is just (3). Now suppose that the quantifier depth of F is $k + 1$ and consists of the formulas

$$\inf_x \psi_1(\bar{x}, x), \ldots, \inf_x \psi_k(\bar{x}, x), \theta_{k+1}(\bar{x}), \ldots, \theta_m(\bar{x})$$

where ψ_i has quantifier depth less then or equal to k for all i, $\bar{x} = (x_1, \ldots, x_n)$ and θ_j is an atomic formula for all j. Consider the set F' which consists of

$$\psi_1(\bar{x}, x_{n+1}), \ldots, \psi_k(\bar{x}, x_{n+k}), \theta_{k+1}(\bar{x}), \ldots, \theta_m(\bar{x}).$$

By induction, we know that Player II has a winning strategy for the game $\mathrm{EF}(M, N, \epsilon, F', n + k)$. Player II plays this winning strategy in order to win $\mathrm{EF}(M, N, \epsilon, F, n)$. To see that this works, suppose that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are the first n plays of the game from M and N respectively. By assumption, we will already have satisfied the winning condition for the atomic formulas θ_j. Now consider $\inf_x \psi_1(\bar{x}, x)$. Suppose that $K = \inf_x \psi^M(a_1, \ldots, a_n, x)$ and we pick $a \in M$ such that $\psi^M(a_1, \ldots, a_n, a) - K < \epsilon/2$. Since Player II is playing according to their winning strategy, we can view a as the play of Player I and pick $b \in N$ so that

$$|\psi^M(a_1, \ldots, a_n, a) - \psi^N(b_1, \ldots, b_n, b)| < \epsilon/2.$$

From this we conclude that $L = \inf_x \psi^N(b_1, \ldots, b_n, x) < K+\epsilon$. On the other hand, if we imagine Player I picking $b \in N$ so that

$$\psi^N(b_1, \ldots, b_n, b) - L < \epsilon/2$$

and Player II responding according to their winning strategy with something in M, we would also conclude that $K < L + \epsilon$. That is to say that $|K-L| < \epsilon$. Since we reserved an new variable for each formula ψ_i there was nothing special about looking at ψ_1 and we conclude that Player II wins. □