Metastability and Model Theory

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The talk will be divided into four sections.

- Section 1 Historical background.
- Section 2 Uniform metastability
- Section 3 Metastable convergence theorems.
- Section 4 Metastability and compactness.

Ongoing research in collaboration with Eduardo Duenez (supported by NSF grant DMS-1500615)

The last result is part of ongoing research with Eduardo Dueñez and Xavier Caicedo (partially supported by NSF grant DMS-11445110).

A *measure preserving system* is a structure of the form $(X, \mathcal{X}, \mu, T)$, where $(X, \mathcal{X}, \mu)$ is a probability space and $T : (X, \mathcal{X}, \mu) \to (X, \mathcal{X}, \mu)$ is a probability space isomorphism. In particular,

- T is invertible,
- T and T^{-1} are measurable,
- $\mu(T^n E) = \mu(T(E))$ for every $E \in \mathcal{X}$ and every integer n.

We are interested in recurrence properties of sets $E \in \mathcal{X}$, or functions $f \in L^p(X, \mathcal{X}, \mu)$.

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Poincaré recurrence theorem

Let$(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let $E \in \mathcal{X}$ be such that $\mu(E) > 0$. Then,

$$\limsup_{n \to \infty} \mu(E \cap T^n E) \geq \mu(E)^2.$$

In particular, $\mu(E \cap T^n E) > 0$, for infinitely many n.

Von Neumann ergodic theorem (1932)

Let $U : H \to H$ be a unitary operator on a separable Hilbert space H. Then the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n v$$

exists for every $v \in H$. Moreover, the limit equals $\pi(v)$, where π is the orthogonal projection from H onto the closed subspace $\{v \in H : Uv = v\}$ consisting of all U-invariant vectors.

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Corollary (Mean ergodic theorem)

Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. Then the limit of averages

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n f$$

exists for every $f \in L^2(X, \mathcal{X}, \mu)$

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1. Replacing $n \mapsto T^n$ by more general group actions, (i.e., $\mathbb{Z}$ by other groups),
2. Considering polynomial, rather than linear actions,
3. Establishing uniform bounds for convergence.

For this talk we will restrict our attention to (3). Let us first start with a well-known example:

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Furstenberg multiple recurrence theorem

Theorem (Furstenberg, 1977)

Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. Then for every set E of positive measure and every positive integer k there exists $n > 0$ such that

$$E \cap T^{-n}E \cap \ldots \cap T^{-(k-1)n}E \neq \emptyset.$$

Uniform Furstenberg multiple recurrence theorem

Theorem (Bergelson, Host, McCutcheon, Parreau, 2000)

For every positive integer k and every $\delta > 0$ there exists $\epsilon(k,\delta) > 0$ with the following property: For every measure-preserving system $(X, \mathcal{X}, \mu, T)$ and every measurable set E with $\mu(E) \geq \delta$,

$$\frac{1}{N}\sum_{k=0}^{N-1} \mu(E \cap T^n E \cap \ldots \cap T^{(k-1)n} E) \geq \epsilon(k,\delta),$$

for all $N \geq 1$.

The norm convergence problem for several commuting transformations

Theorem (Tao, 2007)

If $(X, \mathcal{X}, \mu)$ is a probability space and $T_1, \ldots, T_k : X \to X$ are commuting measure-preserving transformations, then for any bounded measurable functions $f_1, \ldots, f_k : X \to \mathbb{R}$, the multiple averages

$$\frac{1}{N} \sum_{n=0}^{N-1} T_1^n f_1 \ldots T_k^n f_k$$

converge in the $L^2(X)$ norm topology (and hence in probability) as $N \to \infty$.

- The case $k = 1$ is Von Neumann's mean ergodic theorem
- The case $k = 2$: Conze and Lesigne (1983)
- The case for higher l was established by Frantzikinakis and Kra (2005) under additional hypothesis for the operators T_i.
- The case $T_i = T^i$: Host-Kra (2005), Ziegler (2007)

Remark: Tao's argument does not establish a formula for the limit of the sequence of averages. He rather proves that the sequence converges indirectly, by showing that is Cauchy in $L^2(X)$.

For this, he introduces the concept of metastability of sequences and metastable convergence. A crucial component of his proof is his Metastable Dominated Convergence Theorem.

The concept of metastability has been studied from the perspective of computable analysis by Avigad-Dean-Rute, Avigad-Towsner, Kohlenbach, Kohlenbach-Leustea, Kohlenbach-Safarik, Körnlein-Kohlenbach, and Schade-Kohlenbach.

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Walsh's convergence theorem

Theorem (Walsh, 2012)

Let (X, μ) be a measure space with a measure-preserving action of a nilpotent group G. Let $g_1, \ldots, g_k : \mathbb{Z} \to G$ be polynomial sequences in G (i.e. each g_i is of the form $g_i(n) = a_{i,1}^{p_{i,1}(n)} \ldots a_{i,j}^{p_{i,j}(n)}$ for some $a_{i,1}, \ldots, a_{i,j} \in G$ and polynomials $p_{i,1}, \ldots, p_{i,j} : \mathbb{Z} \to \mathbb{Z}$. Then for any $f_1, \ldots, f_k \in L^\infty(X, \mu)$, the averages

$$\frac{1}{N} \sum_{n=1}^{N-1} (g_1(n) f_1) \ldots (g_k(n) f_k)$$

converge in $L^2(X, \mu)$ norm as $N \to \infty$, where $g(n)f(x) := f(g(n)^{-1}x)$.

Remarks:

- Walsh's argument, like Tao's, relies heavily on *metastability*.
- Nilpotence plays a crucial role in Walsh's proof. A key part of his argument uses Leibman's theory of polynomials maps of groups (1998–2002), which relies heavily on nilpotence. Nilpotence is widely regarded as the *non plus ultra* condition ensuring L^2-convergence of multiple ergodic averages.

Definitions: Samplings and metastability rates

Definition

A *sampling* of the totally ordered set $(\mathbb{N}, <)$ is a function

$$\eta : \mathbb{N} \to \mathbb{N}$$

such that $\eta(n) \geq n$ for all $n \in \mathbb{N}$. The set of all samplings of $\mathbb{N}$ will be denoted $\mathrm{Smpl}(\mathbb{N})$.

To each sampling η there corresponds the collection of intervals $[n, \eta(n)] \subset \mathbb{N}$, one for each $n \in \mathbb{N}$.

Definition

A *rate of metastability* is a family

$$E_\bullet = (E_{\epsilon,\eta}) \subset \mathbb{N}$$

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Metastability for sequences with a given rate

Definition ([Tao])

For each sampling η and $\epsilon > 0$ let $E_{\epsilon,\eta} \in \mathbb{N}$ be given.

- A sequence $(a_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is $[\epsilon, \eta]$-*metastable (with bound $E_{\epsilon,\eta}$)* if there exists n (no larger than $E_{\epsilon,\eta}$) such that

$$d(a_m, a_{m'}) \leq \epsilon \qquad \text{for all } m, m' \in [n, \eta(n)].$$

- A sequence is *metastable (with rate $E_\bullet$)* if it is $[\epsilon, \eta]$-metastable (with bound $E_{\epsilon,\eta}$) for every sampling η and all $\epsilon > 0$.

Remarks

- *In general, metastability (with specified rate) is a relaxation of the Cauchy property by restricting to finite sub-tails of (a_n).*
- *When no rates are specified, we have:*
  - *(a_n) is metastable $\Leftrightarrow$ (a_n) is a Cauchy sequence.*

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A Uniform Metastability Principle (UMP)

Proposition (Uniform Metastability Principle [Duenez-I])

Let T be a uniform theory in a signature L such that:

- *L names a sort interpreted as a (discrete) linearly ordered set $(\mathcal{N}, <)$ elementarily extending $(\mathbb{N}, <)$ in models of T, and*
- *L includes a symbol $\mathtt{a}(\cdot)$ for a function $\mathcal{N} \to \mathbb{R}$.*

Then, the following properties are equivalent:

1. *All classical sequences $(a(n) : n \in \mathbb{N})$ obtained by interpreting $\mathtt{a}(\cdot)$ in models of the theory T are Cauchy.*
2. *There exists a collection $E_\bullet = (E_{\epsilon,\eta})$ of metastability rates that applies uniformly to all such sequences.*

Furthermore, when these properties hold, the rate $E_\bullet$ depends only on the theory T.

Some remarks about the UMP

- The UMP follows directly from the compactness theorem for first-order continuous logic.
- It holds for any logic for metric structures that is countably compact.
- $\mathbb{N}$ can be replaced by any directed set (hence it holds for nets, rather than just sequences).
- It essentially states that metastable convergence with a prescribed rate is the only way to capture convergence in first-order continuous logic.

Moreover, the UMP implies the following metatheorem:

> *"Whenever a theorem about convergence of sequences applies to a class of complete metric structures axiomatizable in continuous first-order logic, then the theorem admits a refinement as a statement about uniformly metastable convergence."*

We now switch to applications of this metatheorem.

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DCT structures (Dominated Convergence Theorem)

Let T_{DCT} be the theory (in a suitable signature L) of all structures of the form $\mathcal{M} = (\mathbb{R}, (\mathbb{N}, <), (X, \mu), (\mathcal{L}^\infty(X), \int), \varphi_\bullet)$, where

- $(\mathbb{N}, <)$ is the totally ordered set of natural numbers,
- (X, μ) is a finite measure space,
- $\varphi_\bullet : \mathbb{N} \to B_1(\mathcal{L}^\infty(X))$ is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in the unit ball of $\mathcal{L}^\infty(X)$.

Definition

A *DCT structure* is a countably saturated model $\mathcal{M} = (\mathbb{R}, (\mathcal{N}, <), (X, \mu), (\mathcal{L}_X, \int), \varphi_\bullet)$ of T_{DCT}.

Remark

For all practical purposes (by proxy of a construction analogous to that of Loeb measure in nonstandard analysis) $\aleph_1$-saturation implies that the sort $(X, \mu, \int)$ of a DCT structure is a classical, countably additive probability space and $\int$ is classical integration of functions $f \in \mathcal{L}_X$.

Tao's Metastable Dominated Convergence Theorem

Theorem (Dominated Convergence Theorem (DCT))

Let $\mathcal{M} = (\mathbb{R}, (\mathcal{N}, <), (X, \mu), (\mathcal{L}_X, \int), \varphi_\bullet)$ be a DCT structure. If $(\varphi_n(x))$ is Cauchy for almost all $x \in X$, then $(\int \varphi_n(x) d\mu(x))_{n \in \mathbb{N}}$ is Cauchy.

Since DCT structures are *bona fide* measure spaces endowed with classical integration, the usual proof of DCT applies.

Corollary (Metastable Dominated Convergence Theorem [Tao, 2008)

] For every metastability rate $E_\bullet$ there exists another metastability rate $\widetilde{E_\bullet}$ such that whenever $E_\bullet$ is a metastability rate for the sequences $(\varphi_n(x))$ in $[-1, 1]$, for almost all x in a finite measure space (X, μ), then $\widetilde{E_\bullet}$ is a metastability rate for $\left(\int \varphi_n(x) d\mu(x)\right)$.

Proof.

Extend T_{DCT} to T' by adding the first-order axioms stating that $(\varphi_n(x))$ is $E_\bullet$-metastable for almost all x. Every model $\mathcal{M}$ of T' embeds into a (countably saturated) DCT structure for which DCT holds. By UMP, some metastability rate $\widetilde{E_\bullet}$ must apply to all sequences $\left(\int \varphi_n(x) d\mu(x)\right)$. □

Tao's Metastable Dominated Convergence Theorem

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Metastable Mean Ergodic Theorem

Theorem (Mean Ergodic Theorem (von Neumann 1932))

Given a unitary transformation U on a Hilbert space H and a point $x \in H$, the sequence

$$\mathrm{AV}_N(x) = \frac{1}{n} \sum_{k=0}^{N-1} U^n x \qquad (n \in \mathbb{N})$$

of ergodic averages converges as $N \to \infty$.

Corollary (Metastable Mean Ergodic Theorem)

There exists a universal metastability rate $E_\bullet$ such that the sequence of ergodic averages $(\mathrm{AV}_N(x))$ of any point x in the unit ball of any Hilbert space H under any unitary operator U on H is $E_\bullet$-metastable.

Ergodic almost-everywhere convergence

Proposition (Metastable Birkhoff ergodic theorem)

For every $\delta > 0$ there exists a rate $E_\bullet^{(\delta)}$ such that if T is measure-preserving on a probability space (X, μ), then for every measurable f such that $\|f\|_\infty \leq 1$ there exists measurable $Y \subset X$ such that

- $\mu(X \setminus Y) \leq \delta$, *and*
- $\left(\frac{1}{n}\sum_{j<n} f \circ T^j(y)\right)_{n\in\mathbb{N}}$ *converges pointwise with metastable rate $E_\bullet^{(\delta)}$ for all $y \in Y$.*

Remarks

- *Apart from the dependence on δ, the rate $E_\bullet^{(\delta)}$ is completely universal (independent of (X, μ, T)).*
  - *This should be contrasted with the almost-uniform convergence implied by Egorov's theorem, where the rates of uniform convergence depend not only on δ but also on the transformation T.*
- *In this formulation, it is necessary to impose a bound on $\|f\|_\infty$ (not merely on $\|f\|_1$).*

Metastability and compactness

As we have seen the Uniform Metastability Principle (UMP) is a consequence of the compactness of first-order continuous logic. In fact, it holds in any logic for metric structures that satisfies countable compactness.

Question (Tao)

Is there a precise connection between the metastability and compactness?

PC-classes

Let $\mathscr{C}$ be a class of L-structures.

$\mathscr{C}$ is said to be a *PC-class* if $\mathscr{C}$ can be axiomatized by a single sentence in some signature $L' \supseteq L$.

Equivalently, $\mathscr{C}$ is a PC-class if $\mathscr{C}$ can be axiomatized by an existential second-order L-sentence.

This definition applies to any logic: Given a logic $\mathscr{L}$ one can consider the PC-classes of $\mathscr{L}$.

Fact

If $\mathbb{L}$ is a compact logic, then the logic of existential second-order sentences of $\mathscr{L}$ inherits compactness from $\mathbb{L}$.

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An L-structure $\mathcal{M}$ is *RPC_Δ-characterizable* if there exists a predicate R and an existential second-order $L \cup \{R\}$-theory T such that the restriction of any model of T to the predicate R is isomorphic to $\mathcal{M}$.

Theorem (X. Caicedo, E. Duenez, I)

Let $\mathscr{L}$ be a logic that is not countably compact. If $\mathcal{M}$ is any a metric structure of cardinality less than the first measurable cardinal, then $(\mathcal{M}, a)_{a\in\mathcal{M}}$ is RPC_{Δ}-caracterizable in $\mathscr{L}$.

This allows us to show that

$$\text{Uniform Metastability Principle} \quad \Leftrightarrow \quad \text{Compactness.}$$

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More precisely, we have:

Corollary

Let $\mathscr{L}$ be a logic.

1. *If $\mathscr{L}$ is countably compact, then $\mathscr{L}$ satisfies UMP.*
2. *If $\mathscr{L}$ is not countably compact, then UMP fails for $\mathrm{RPC}_\Delta(\mathscr{L})$.*

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