Continuous Model Theory

Lecture 1: Ultraproducts and Continuous Logic

Bradd Hart

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Overview

- Lecture 1: Ultraproducts and continuous logic
- Lecture 2: Some of the basics and II_1 factors
- Lecture 3: Definable sets and nuclearity
- Lecture 4: Ultrapowers of separable II_1 factors

See www.math.mcmaster.ca/∼bradd/IrvineCMT.html for the slides and links to other resources.

Filters and Ultrafilters

Definition

If X is a set and $F \subseteq \mathcal{P}(X)$ then F is said to be a filter if

- $\emptyset \notin F$,
- if $A, B \in F$ then $A \cap B \in F$, and
- if $A \in F$ and $A \subseteq B \subseteq X$ then $B \in F$.

Lemma

$G \subseteq \mathcal{P}(X)$ is contained in a filter iff G has the finite intersection property i.e. for every finite $G_0 \subseteq G$, $\bigcap G_0 \neq \emptyset$.

Definition

An ultrafilter on X is a filter F such that for every $A \subseteq X$, either $A \in F$ or $X \setminus A \in F$.

Lemma

- *If F is a filter on X then F is an ultrafilter iff it is a maximal filter.*
- *Any filter on X can be extended to an ultrafilter.*

Ultralimits

Now suppose $\mathcal{U}$ is an ultrafilter on a set I and $\bar{r} = \langle r_i : i \in I \rangle$ is an I-indexed family of real numbers. We define the ultralimit of $\bar{r}$ with respect to $\mathcal{U}$ as follows:

$$\lim_{i \to \mathcal{U}} r_i = r \text{ iff for every } \epsilon > 0, \{i \in I : |r - r_i| < \epsilon\} \in \mathcal{U}.$$

Lemma

If $\bar{r}$ is bounded then

- $\lim_{i \to \mathcal{U}} r_i$ *exists and is unique.*
- $\lim_{i \to \mathcal{U}} r_i = \inf\{B : \{i \in I : r_i < B\} \in \mathcal{U}\}$.
- $\lim_{i \to \mathcal{U}} r_i = \sup\{B : \{i \in I : r_i > B\} \in \mathcal{U}\}$.

Ultraproducts of metric spaces

Fix an index set I, an ultrafilter $\mathcal{U}$ on I and uniformly bounded metric spaces (X_i, d_i) for $i \in I$ i.e. there is some B so that for all i and all $x, y \in X_i$, $d_i(x, y) \leq B$. Define d on $\prod_{i \in I} X_i$ as follows:

$$d(\overline{x}, \overline{y}) = \lim_{i \to \mathcal{U}} d_i(x_i, y_i)$$

Lemma

d is a pseudo-metric on $\prod_{i \in I} X_i$.

Definition

The ultraproduct of the X_i's with respect to $\mathcal{U}$, $\prod_{\mathcal{U}} X_i$, is the metric space obtained by quotienting $\prod_{i \in I} X_i$ by d. If all the X_i's are equal to a fixed X we will often write $X^{\mathcal{U}}$ for this ultraproduct and call it the ultrapower.

Metric structures

- We want to add more structure to a (bounded) metric space; for now let's consider a single additional function f.
- So we will have a bounded metric space (X,d) and a function f say of one variable. We do want that the ultraproduct of these structures is still a structure of the same kind. So how do we define f on the ultrapower of X?
- f must be continuous!
- f must be uniformly continuous!
- There is nothing special about one variable; these arguments apply to functions of many variables.

Metric structures cont'd

- What about relations? Imagine that we have a one-variable relation R (taking values somewhere) on a metric space and we want to make sense of it in the ultrapower.
- Its range must be compact and R must be uniformly continuous.
- There is really no loss in assume that the range of R is $[0,1]$ or some other compact interval in the reals.
- Again there is nothing special about one-variable; we can have relations of many variables.

The language of a metric structure

A language $\mathcal{L}$ will consist of

- a set $\mathcal{S}$ called sorts;
- $\mathcal{F}$, a family of function symbols. For each $f \in \mathcal{F}$ we specify the domain and range of f: $dom(f) = \prod_{i=1}^{n} S_i$ where $S_1, \dots, S_n \in \mathcal{S}$ and $rng(f) = S$ where $S \in \mathcal{S}$. Moreover, we also specify a continuity modulus. That is, we are given $\delta^f : [0,1] \to [0,1]$; and
- $\mathcal{R}$, a family of relation symbols. For each $R \in \mathcal{R}$ we are given the domain $dom(R) = \prod_{i=1}^{n} S_i$ where $S_1, \dots, S_n \in \mathcal{S}$ and the $rng(R) = K_R$ for some closed interval K_R. Moreover, for each i, we specify a continuity modulus $\delta^R : [0,1] \to [0,1]$.
- For each $S \in \mathcal{S}$, we have one special relation symbol d_S with domain $S \times S$ and range of the form $[0, B_s]$. It's continuity moduli is the identity function.

Definition of a metric structure

A metric structure $\mathcal{M}$ *interprets* a language $\mathcal{L}$; it will consist of

- an $\mathcal{S}$-indexed family of complete bounded metric spaces $(S^{\mathcal{M}}, d_S^{\mathcal{M}})$ with bound B_S for $S \in \mathcal{S}$;
- a family of functions $f^{\mathcal{M}}$ for every $f \in \mathcal{F}$ such that $\mathrm{dom}(f^{\mathcal{M}}) = \bar{S}^M = \prod_{i=1}^n S_i^{\mathcal{M}}$ where $\mathrm{dom}(f) = \prod_{i=1}^n S_i$ and $\mathrm{rng}(f^{\mathcal{M}}) = S^{\mathcal{M}}$ where $\mathrm{rng}(f) = S$. $f^{\mathcal{M}}$ is uniformly continuous as specified by the uniform continuity moduli associated to f; that is if for every $\epsilon > 0$ and $\delta = \delta^f(\epsilon)$, if $\bar{a}, \bar{b} \in \bar{S}^M$ and $d_{\bar{S}^M}(\bar{a}, \bar{b}) < \delta$ then
$$d_S^{\mathcal{M}}(f^{\mathcal{M}}(\bar{a}), f^{\mathcal{M}}(\bar{b})) \leq \epsilon;$$
- a family of relations $R^{\mathcal{M}}$ for every $R \in \mathcal{R}$ such that $\mathrm{dom}(R^{\mathcal{M}}) = \prod_{i=1}^n S_i^{\mathcal{M}}$ where $\mathrm{dom}(R) = \prod_{i=1}^n S_i$ and $\mathrm{rng}(R^{\mathcal{M}}) = K_R$. $R^{\mathcal{M}}$ is uniformly continuous as specified by the uniform continuity moduli associated to R as above.

Examples of metric structures

Some simple examples:

- Any complete bounded metric space (X, d). This has the empty family of functions and relations although we often count the metric as a relation (why is it uniformly continuous?)
- Any ordinary first order structure M with some collection of functions and relations. To see this as a metric structure, we put the discrete $\{0, 1\}$-valued metric on M to make it a bounded metric space. All functions become uniformly continuous. Relations which are usually thought of as subsets of M^n become $\{0, 1\}$-valued functions - again they are uniformly continuous.

C^*-algebras as metric structures

- Fix a C^*-algebra A. We introduce sorts B_n for each $n \in \mathbb{N}$.
- Let $B_n(A)$ be the ball of radius n centered at 0; $B_n(A)$ is a bounded complete metric space with respect to the metric induced by the operator norm.
- There are inclusion maps between B_n and B_m if $n \leq m$.
- 0 is a constant (our functions can have arity 0!) in B_1 and if A is unital then 1 is in B_1 as well.
- For scalars λ and for every n, there is a unary function λ_n which is scalar multiplication by λ on B_n; this function has range in B_m where m is the least integer greater than or equal to $n|\lambda|$.
- The operation of addition has to be similarly divided up: for $m, n \in \mathbb{N}$, there is an operation $+_{m,n}$ which takes $B_m \times B_n$ to B_{m+n}.
- Multiplication is similarly divided up depending on the balls as is the adjoint operation.

C^*-algebras, cont'd

- The metric is given via the operator norm as $d_n(x, y) = \|x - y\|$ on B_n.
- So formally a C^*-algebra will be thought of as a metric structure by considering
  - The family of bounded metric spaces B_n for all $n \in \mathbb{N}$ with metrics d_n, as well as
  - the family of functions 0, λ_n for scalars λ and $n \in \mathbb{N}$, inclusion maps between the sorts, and $+_{m,n}$, $\cdot_{m,n}$ and $^*_n$ for all $m, n \in \mathbb{N}$.
- It is routine to check that all of these functions are uniformly continuous (the only issue is multiplication and this holds because we have restricted the norm).
- The sorts are complete since C^*-algebras are complete.
- In what sense do these metric structures capture the class of C^*-algebras? We need to introduce a little more model theory to answer this question.

Ultraproducts of metric structures

Fix a language $\mathcal{L}$, an index set I, an ultrafilter $\mathcal{U}$ on I and $\mathcal{L}$-structures $\mathcal{M}_i$ for $i \in I$.

Definition

The ultraproduct of the $\mathcal{M}_i$'s with respect to $\mathcal{U}$, $\prod_{\mathcal{U}} \mathcal{M}_i$ is the $\mathcal{L}$-structure $\mathcal{M}$ defined as follows:

1. for every sort S, $S^{\mathcal{M}} = \prod_{\mathcal{U}} S^{\mathcal{M}_i}$ with metric $d_S^{\mathcal{M}} = \lim_{i \to \mathcal{U}} d_S^{\mathcal{M}_i}$,
2. for every function symbol f with range S
$$f^{\mathcal{M}}(\bar{x}) = \langle f^{\mathcal{M}_i}(\bar{x}_i) : i \in I \rangle / d_S^{\mathcal{M}}, \text{ and}$$
3. for every relation symbol R,
$$R^{\mathcal{M}} = \lim_{i \to \mathcal{U}} R^{\mathcal{M}_i}.$$

If all of the $\mathcal{M}_i$'s are a fixed $\mathcal{N}$, we call this the ultrapower and write $\mathcal{N}^{\mathcal{U}}$.

C^*-algebraic ultraproducts

- A dead give-away that model theory is involved is that operator algebraists are using ultraproducts.
- Suppose A_i are C^*-algebras for all $i \in I$ and that $\mathcal{U}$ is an ultrafilter on I. Consider the bounded product

$$\prod^b A_i := \{\bar{a} \in \prod A_i : \lim_{i \to \mathcal{U}} \|a_i\| < \infty\}$$

  and the two-sided ideal $c_{\mathcal{U}}$

$$\{\bar{a} \in \prod^b A_i : \lim_{i \to \mathcal{U}} \|a_i\| = 0\}.$$

  The ultraproduct, $\prod_{\mathcal{U}} A_i$ is defined as $\prod^b A_i / c_{\mathcal{U}}$.
- One checks that multiplication, addition and the adjoint are well-defined coordinatewise modulo $c_{\mathcal{U}}$.

Terms and formulas

For a language $\mathcal{L}$, terms are defined inductively from function symbols and variables by composition exactly as in discrete logic. The only wrinkle is that one needs to keep track of the continuity modulus of terms determined by composition. Formulas are defined inductively.

Definition

- Suppose R is a relation symbol in $\mathcal{L}$ with $\mathrm{dom}(R) = \prod_{i=1}^{n} S_i$ and $\mathrm{rng}(R) = K_R$, and τ_i are terms where $\mathrm{rng}(\tau_i) = S_i$ for all i. Then $R(\tau_1, \ldots, \tau_n)$ is a formula. The domain, range and continuity moduli are those obtained by composition.
- Suppose $\varphi_i(\overline{x})$ is a formula with range K_{φ_i} for all $i \leq n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Then $f(\varphi_1, \ldots, \varphi_n)$ is a formula with range $f(\prod_{i=1}^{n} K_{\varphi_i})$ and domain and continuity moduli determined by composition.
- If φ is a formula and x is a variable then $\sup_x \varphi$ and $\inf_x \varphi$ are both formulas. The sort of x is removed from the domain; the range and continuity moduli for the remaining variables stay the same.

Interpretations

Fix a metric structure $\mathcal{M}$ for a language $\mathcal{L}$.

- Terms are interpreted by composition inductively as usual.
- For the formula $R(\tau_1(\overline{x}), \ldots, \tau_n(\overline{x}))$ where R is a relation in $\mathcal{L}$ and $\tau_1, \ldots, \tau_n$ are terms, its interpretation is given, for every appropriate $\overline{a} \in \mathcal{M}$, by

$$R^M(\tau_1^{\mathcal{M}}(\overline{a}), \ldots, \tau_n^{\mathcal{M}}(\overline{a}))$$

- If $\varphi_i(\overline{x})$ is a formula for all $i \leq n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function then if ψ is the formula $f(\varphi_1, \ldots, \varphi_n)$ then $\psi^M := f(\varphi_1^{\mathcal{M}}, \ldots, \varphi_n^{\mathcal{M}})$.
- Suppose $\varphi(x, \overline{y})$ is a formula and $\overline{a} \in \mathcal{M}$ is a tuple appropriate for the variables $\overline{y}$ and x is of sort S. Then

$$(\sup_x \varphi(x, \overline{a}))^{\mathcal{M}} := \sup\{\varphi^{\mathcal{M}}(b, \overline{a}) : b \in S^{\mathcal{M}}\}$$

and

$$(\inf_x \varphi(x, \overline{a}))^{\mathcal{M}} := \inf\{\varphi^{\mathcal{M}}(b, \overline{a}) : b \in S^{\mathcal{M}}\}.$$

Basic properties

In an $\mathcal{L}$-structure $\mathcal{M}$

- the interpretations of terms in $\mathcal{M}$ are uniformly continuous functions with continuity modulus specified by the definition of the term, and
- all formulas when interpreted in $\mathcal{M}$, define uniformly continuous functions with domains, range and continuity modulus specified by the definition of the formula.
- A sentence is a formula with no free variables. Any sentence in $\mathcal{L}$ takes on a value in an $\mathcal{L}$-structure in a compact interval specified by $\mathcal{L}$ and this interval is independent of the given structure.
- An $\mathcal{L}$-structure $\mathcal{M}$ satisfies a sentence φ if $\varphi^{\mathcal{M}} = 0$.
- If T is a set of $\mathcal{L}$-sentences then the class of all $\mathcal{L}$-structures which satisfy all sentences in T is called Mod(T), the models of T. T is called a set of axioms for this class. A class of $\mathcal{L}$-structures is called an elementary class if it is Mod(T) for some T.

Łoś' Theorem

Theorem

Suppose $\mathcal{M}_i$ are $\mathcal{L}$-structures for all $i \in I, \mathcal{U}$ is an ultrafilter on I, $\varphi(\overline{x})$ is an $\mathcal{L}$-formula and $\overline{a} \in \mathcal{M} := \prod_{i \in I} \mathcal{M}_i/\mathcal{U}$ then

$$\varphi^{\mathcal{M}}(\overline{a}) = \lim_{i \to U} \varphi^{\mathcal{M}_i}(\overline{a}_i).$$

Corollary

Elementary classes are closed under ultraproducts.

Axioms for C^*-algebras

- There are many universal axioms expressing that a C^*-algebra is a Banach *-algebra. These involve saying that two terms are equal or that some norm or other is equal to or less than something else.
- For instance, to say $\tau(\bar{x}) = \sigma(\bar{x})$ we need to write $\sup_{\bar{x}} d(\tau(\bar{x}), \sigma(\bar{x}))$ which is awful so we write the first and mean the second.
- For the metric, we have $d(x, 0) = \|x\|$ and $d(x, y) = \|x - y\|$. One can now write out the axioms for a Banach space.
- Include $\|x^*\| = \|x\|$ and the C^*-identity, $\|x^* x\| = \|x\|^2$.
- Now comes the fussy bits about using balls: we have

$$\sup_{x \in B_1} \|x\| \leq 1 \text{ and } \sup_{x \in B_n} \min\{1 \dot{-} \|x\|, \inf_{y \in B_1} \|x - y\|\}.$$

Theorem

These axioms completely capture the class of metric structures associated to C^*-algebras.