Continuous Model Theory

Lecture 2: Some of the basics and II_1 factors

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Satisfiability

Definition

- We say a set of sentences Σ in a language $\mathcal{L}$ is satisfied if there is an $\mathcal{L}$-structure $\mathcal{M}$ such that for every sentence in Σ holds in $\mathcal{M}$ i.e. for every $\varphi \in \Sigma$, $\varphi^{\mathcal{M}} = 0$.
- We say such a Σ is finitely satisfied if every finite subset of Σ is satisfied.
- For a set of sentence Σ and $\epsilon > 0$, the ϵ-approximation of Σ is

$$\{|\varphi| \dot{-} \epsilon : \varphi \in \Sigma\}$$

- Σ is approximately finitely satisfied if for every $\epsilon > 0$, the ϵ-approximation of Σ is finitely satisfiable.

Compactness

Theorem

TFAE for a set of sentences Σ in a language $\mathcal{L}$

- *Σ is satisfiable.*
- *Σ is finitely satisfiable.*
- *Σ is approximately finitely satisfiable.*

A metric on formulas

Fix a language $\mathcal{L}$ and fix a tuple of variables $\overline{x}$ from a sequence of sorts $\overline{S}$. We define a pseudo-metric on the formulas with free variables $\overline{x}$ as follows: we define the distance between $\varphi(\overline{x})$ and $\psi(\overline{x})$ to be

$$\sup\{|\varphi^{\mathcal{M}}(\overline{a}) - \psi^{\mathcal{M}}(\overline{a})| : \mathcal{M}, \text{ an } \mathcal{L}\text{-structure, and } \overline{a} \in \mathcal{M}\}$$

We will call this space $\mathcal{F}_{\overline{S}}$. This can also be relativized to all structures satisfying a fixed theory.

Density character

Definition

We say that the density character of a topological space X is the infinum of the cardinality of a dense subset of X. We will write $\chi(X)$ for the density character of X.

Note: An infinite separable space has countable density character.

Proposition

If $\mathcal{L}$ is countable i.e. there are only countably many relation and function symbols, then for any tuple of sorts $\overline{S}$, $\mathcal{F}_{\overline{S}}$ is separable.

Notation: $\chi(\mathcal{L})$ will mean $\sum_{\overline{S}} \chi(\mathcal{F}_{\overline{S}})$.

Embeddings and elementary submodels

- Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures such that the universe of $\mathcal{M}$ is a closed subset of $\mathcal{N}$. $\mathcal{M}$ is called a submodel if all functions and relations from $\mathcal{L}$ on $\mathcal{M}$ are the restriction of those from $\mathcal{N}$. We write $\mathcal{M} \subseteq \mathcal{N}$.
- For $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M}$ is an *elementary* submodel if, for every $\mathcal{L}$-formula $\varphi(\overline{x})$ and every $\overline{a} \in \mathcal{M}$, $\varphi^{\mathcal{M}}(\overline{a}) = \varphi^{\mathcal{N}}(\overline{a})$. We write $\mathcal{M} \prec \mathcal{N}$.
- An embedding between metric structures is a map which preserves the functions and relations. An embedding is elementary if its image is an elementary submodel of the range.

Notice that by Łoś' Theorem, any metric structure $\mathcal{M}$ embeds elementarily into its ultrapower $\mathcal{M}^{\mathcal{U}}$ for any ultrafilter $\mathcal{U}$ via the diagonal embedding.

Downward Löwenheim-Skolem

Proposition (Tarski-Vaught)

If $\mathcal{M} \subseteq \mathcal{N}$ then $\mathcal{M}$ is an elementary submodel if for every formula $\varphi(x, \overline{y})$, $r \in \mathbb{R}$ and $\overline{a} \in \mathcal{M}$, if $(\inf_x \varphi(x, \overline{a}))^{\mathcal{N}} < r$ then there is $b \in \mathcal{M}$ such that $(\varphi(b, \overline{a}))^{\mathcal{N}} < r$.

Theorem (DLS)

Suppose that $\mathcal{N}$ is an $\mathcal{L}$-structure and A is a subset of $\mathcal{N}$. Then there is an elementary submodel $\mathcal{M} \subseteq \mathcal{N}$ such that

1. *A is contained in $\mathcal{M}$ and*
2. *for every sort S,*

$$\chi(S^{\mathcal{M}}) \leq \chi(\mathcal{L}) + \chi(A)$$

Some abstract model theory

Theorem

For a class of $\mathcal{L}$-structures $\mathcal{C}$, TFAE

1. *$\mathcal{C}$ is an elementary class.*
2. *$\mathcal{C}$ is closed under isomorphisms, ultraproducts and elementary submodels.*
3. *$\mathcal{C}$ is closed under isomorphisms, ultraproducts and ultraroots.*

Tracial ultraproducts

Suppose M_i are von Neumann algebras with faithful normal traces τ_i for all $i \in I$ and $\mathcal{U}$ is an ultrafilter on I. Again, the bounded product is

$$\prod^b M_i := \{\bar{a} \in \prod M_i : \lim_{i \to \mathcal{U}} \|a_i\| < \infty\}$$

and now consider the two-sided ideal

$$c_{\mathcal{U}} = \{\bar{a} \in \prod^b M_i : \lim_{i \to \mathcal{U}} \tau_i(a_i^* a_i) = 0\}.$$

The ultraproduct, $\prod_{\mathcal{U}} M_i$, is defined as $\prod^b M_i / c_{\mathcal{U}}$. It is a tracial von Neumann algebra with the trace given by $\tau(\bar{x}) = \lim_{i \to \mathcal{U}} \tau_i(x_i)$.

Tracial vNas as metric structures

- The tracial ultraproducts actually guide us in figuring out how to see tracial von Neumann algebras as metric structures.
- As with C*-algebras, we introduce sorts for the balls of operator norm n for each $n \in \mathbb{N}$. The big difference is that the operator norm will **NOT** be in the language.
- The basic functions are again considered as partitioned across the sorts together with the necessary inclusion maps. We also have the trace which formally we have to break into its real and imaginary parts.
- Remember that on a tracial von Neumann algebra, there is a faithful normal trace and we can define a norm, the 2-norm, by taking $\|x\|_2 = \sqrt{\tau(x^* x)}$.
- The metric on each ball is induced by the 2-norm; it is complete on each ball. It is critical that the 2-norm is restricted to a bounded ball.

Do tracial vNas form an elementary class?

- Let $\mathcal{K}$ be the class of metric structures arising from tracial von Neumann algebras as on the previous slide. Is $\mathcal{K}$ an elementary class? Let's check this semantically.
- For both ultraproducts of tracial von Neumann algebras and ultraroots, one must see that the resulting structures are tracial von Neumann algebras in their own right. So suppose that A is either an ultraproduct or ultraroot of something from $\mathcal{K}$.
- Any such A can be identified with a *-algebra with a faithful trace τ by taking the direct limit of its sorts.
- Using the trace, we create an inner product on A by $\langle x, y \rangle = tr(y^* x)$. Let H be the associated Hilbert space.
- A acts on H by left multiplication; this is the standard representation of A and so we can think of A as a *-subalgebra of $B(H)$.
- The fact that the unit ball is closed in the sense of the 2-norm implies that it is closed in the weak topology on $B(H)$.
- Enter Kaplansky density.

Kaplansky density

Theorem (Kaplansky)

*Suppose that A is a *-subalgebra of $B(H)$ then*

$$(\overline{A}^{w})_1 = \overline{A_1}^{w}.$$

*That is, the process of taking the weak closure and the unit ball commute for *-subalgebras.*

$\mathcal{K}$ and the class of tracial von Neumann algebras are naturally bijective is taken care of using the same trick as with C^*-algebras.

II_1 factors

- A von Neumann algebra whose centre is $\mathbb{C}$ is called a factor.
- A tracial factor is type I if all its projections have rational trace and is type II_1 if the range of the trace on projections is [0,1].
- $\mathcal{R}$, $\mathcal{R}^{\mathcal{U}}$, $\prod_{\mathcal{U}} M_n(\mathbb{C})$ and $L(F_n)$ are all II_1 factors.

Theorem (FHS)

The class of II_1 factors viewed as metric structures is an elementary class.

Proof that II_1 factors forms an elementary class

- II_1 factors form a subclass of tracial von Neumann algebras. It suffices then to see that being type II_1 factor is preserved under ultraproducts and elementary submodels.
- To see this, note the following inequality that holds in II_1 factor $\mathcal{M}$: for all $x \in \mathcal{M}_1$,

$$\|x - \tau(x) \cdot 1\|_2 \leq \sup_{y \in \mathcal{M}_1} \|[x, y]\|_2.$$

  Any structure in which this sentence holds is in fact a factor.
- Once we know that tracial factors form an elementary class, we can use the range of the trace to see that both an ultraproduct of II_1 factors is type II_1 and the same is true of an elementary submodel.