Continuous Model Theory

Lecture 3: Definability and Nuclearity

Bradd Hart

Sept. 21, 2017

What is a formula?

Relative to a theory T, define a pseudo-norm on $\mathcal{F}_{\bar{x}}$ by

$$\|\varphi(\bar{x})\| := \sup\{|\varphi^{\mathcal{M}}(\bar{a})| : \bar{a} \in \mathcal{M}\}.$$

This puts a normed linear structure on $\mathcal{F}_{\bar{x}}$.

Definition

A Cauchy sequence of formulas $\overline{\varphi}$ in $\mathcal{F}_{\overline{x}}$ will be called a definable predicate and interpreted in an $\mathcal{L}$-structure $\mathcal{M}$ by

$$\overline{\varphi}^{\mathcal{M}}(\overline{a}) = \lim_{n\to\infty} \varphi_n^{\mathcal{M}}(\overline{a}).$$

Of course what we are doing is extending the notion of formula to the Banach space generated by $\mathcal{F}_{\overline{x}}$.

Zero sets and distance predicates

Fix a theory T in a language $\mathcal{L}$ and a model $\mathcal{M}$ of T.

- For a definable predicate $\varphi(\bar{x})$, the zero set of φ in $\mathcal{M}$ is

$$Z(\varphi^{\mathcal{M}}) := \{\bar{a} \in \mathcal{M} : \varphi^{\mathcal{M}}(\bar{a}) = 0\}.$$

- If X is a non-empty closed subset of some product of sorts in $\mathcal{M}$ we call $P(x) = d(x, X) = \inf\{d(x, y) : y \in X\}$ the distance predicate for X.

Definable sets

We introduce the category Met where the objects are bounded metric spaces and the morphisms are isometries.

Definition

Suppose we have a theory T in a language $\mathcal{L}$ and S_i for $i \leq n$ are sorts in $\mathcal{L}$. We call a functor

$$X : \mathrm{Mod}(T) \to \mathrm{Met}$$

a uniform assignment relative to T if for every model $\mathcal{M}$ of T, $X(\mathcal{M})$ is a closed subset of $\prod_{j=1}^{m} S_j^{\mathcal{M}}$ and X is just restriction on morphisms. This assignment is called a *definable set* if, for all formulas $\psi(\bar{x}, \bar{y})$, the functions defined for all $\mathcal{M}$, models of T, by

$$\sup_{\bar{x} \in X(\mathcal{M})} \psi^{\mathcal{M}}(\bar{x}, \bar{y}) \quad \text{and} \quad \inf_{\bar{x} \in X(\mathcal{M})} \psi^{\mathcal{M}}(\bar{x}, \bar{y})$$

are definable predicates for T.

Examples

- Sorts are definable sets as are products of sorts.
- Products of definable sets are definable.
- The range of a term is definable.
- For example, in the operator algebra setting, the set of self-adjoint elements in any algebra is definable as is the set of positive elements: $(x + x^*)/2$ and x^*x are the two terms in question.

Critical remarks about definable sets

- A natural source of uniform assignments is the zero-set of any definable predicate.
- If an assignment is a definable set then it is the assignment arising from the zero-set of some definable predicate. Just choose $\psi(\bar{x}, \bar{y}) := d(\bar{x}, \bar{y})$ and parse $\inf_{\bar{x} \in X(\mathcal{M})} \psi(\bar{x}, \bar{y})$.
- The definition of definable set could be read

  "Definable sets are those sets you can quantify over."

  Notice in the discrete case, you can quantify over the solution set of any formula.
- There are lots of zero sets which are NOT definable sets; we will see some in a few slides.

A useful lemma

Lemma (MTFMS, 2.10)

Suppose that $F, G : X \to [0,1]$ *are functions such that*

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in X \ (F(x) \leq \delta \implies G(x) \leq \epsilon)$$

Then there exists an increasing, continuous function $\alpha : [0,1] \to [0,1]$ *such that* $\alpha(0) = 0$ *and*

$$\forall x \in X \ (G(x) \leq \alpha(F(x))).$$

A second characterization of definable sets

Theorem

Suppose that X is a uniform assignment relative to a theory T. Then the following are equivalent:

1. *This assignment is a definable set.*
2. *The distance predicate $d(\bar{x}, X)$ is a definable predicate for T.*
3. *There is a definable predicate $\varphi(\bar{x})$ so that for any model of T, $\mathcal{M}$, $Z(\varphi^{\mathcal{M}}) \subseteq X(\mathcal{M})$ and for every $\epsilon > 0$ there is a $\delta > 0$,*

$$\text{if } \varphi^{\mathcal{M}}(\bar{a}) < \delta \text{ then } d(\bar{a}, X(\mathcal{M})) \leq \epsilon.$$

A proof

For (2) implies (1), fix a formula ψ. It is uniformly continuous so using MTFMS 2.10, we can find continuous α such that for all $\bar{x}, \bar{y}$ and $\bar{z}$

$$|\psi(\bar{x}, \bar{z}) - \psi(\bar{y}, \bar{z})| \leq \alpha(d(\bar{x}, \bar{y})).$$

Consider

$$\inf_{\bar{z}}(\psi(\bar{x}, \bar{z}) + \alpha(d(\bar{z}, X))) \text{ and } \inf_{\bar{z} \in X} \psi(\bar{x}, \bar{z}).$$

The claim is that these are equal and the first is a definable predicate.

Examples

- Definable sets are closed under unions.
- In the operator algebra setting, the set of projections in a given algebra is a definable set.
- This is related to the notion of weakly stable relations: a definable predicate $\varphi(\bar{x})$ is said to be weakly stable (or be a weakly stable relation) relative to the theory T if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\mathcal{M}$ is a model of T and $\varphi^{\mathcal{M}}(\bar{a}) < \delta$ for some $\bar{a} \in \mathcal{M}$ then there is $\bar{b} \in \mathcal{M}$ such that $\varphi(\bar{b}) = 0$ and $d(\bar{a}, \bar{b}) \leq \epsilon$.
- The zero set of a weakly stable relation is clearly a definable set.
- There are many examples of weakly stable relations in the operator algebraic literature; for instance, that an n^2-tuple x_{ij} for $i, j \leq n$ are the matrix units of a unital copy of $M_n(\mathbb{C})$.

A third characterization of definable sets

Theorem

Suppose that X is a uniform assignment relative to a theory T. Then the following are equivalent:

1. *This assignment is a definable set.*
2. *For all sets I, ultrafilters $\mathcal{U}$ on I and models of T, $\mathcal{M}_i$ for $i \in I$, if $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$ then*

$$X(\mathcal{M}) = \prod_{\mathcal{U}} X(\mathcal{M}_i).$$

Examples of definable sets

- The set of normal elements in $B(H)$ is not definable (or more correctly the set of normal elements is not a definable set relative to the theory of $B(H)$ or the theory of C^*-algebras.)
- The ball of radius 1 around a point in the ball of radius 1 in a Hilbert space. Far more generally, if the underlying metric space has geodesics then balls will be definable sets.
- Ultrametrics give examples that are not definable. Here is a toy example: on the interval $[0, 2]$ define the metric d

$$d(x, y) = \begin{cases} \max\{x, y\} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

  If we fix 0 as a constant than the zero-set of $d(0, x) \leq 1$ doesn't survive ultrapowers. These types of examples arise naturally from metrics associated to certain valuations.

Beth definability

Theorem

Suppose that $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2$ are languages, $\mathcal{L}_0$ and $\mathcal{L}_1$ have the same sorts and T is an $\mathcal{L}_2$-theory. Further suppose whenever $\mathcal{M}$ is a model of T, $\mathcal{M}_0 = \mathcal{M} \upharpoonright_{\mathcal{L}_0}$ and $\mathcal{U}$ is an ultrafilter then:

if $f : \mathcal{M}_0 \to \mathcal{M}_0^{\mathcal{U}}$ is an $\mathcal{L}_0$-elementary map then f is also an $\mathcal{L}_1$-map.

Then every $\mathcal{L}_1$-formula is T-equivalent to a definable predicate in $\mathcal{L}_0$.

Corollary

Suppose $\mathcal{L}_0 \subseteq \mathcal{L}_1$ are two languages with the same sorts, T_i is a theory in $\mathcal{L}_i$ for $i = 0, 1$ and the forgetful functor

$$F : Mod(T_1) \to Mod(T_0)$$

given by restriction to $\mathcal{L}_0$ is an equivalence of categories. Then every $\mathcal{L}_1$-formula is T_1-equivalent to a definable predicate in $\mathcal{L}_0$.

Model theoretic characterization of nuclear algebras

Consider, for $k, n \in \mathbb{N}$, the predicate defined on A_1^k by

$$R_n^k(\bar{a}) = \inf_{\varphi,\psi} \|\bar{a} - \psi(\varphi(\bar{a}))\|$$

where $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ range over cpc maps.

Theorem (FHLRTVW)

R_n^k is a definable predicate in the language of C^*-algebras.

Corollary

A C^*-algebra is nuclear if it satisfies, for all k,

$$\sup_{\bar{x}} \inf_n R_n^k(\bar{x}).$$

A sketch of the proof

- Fix n and k and start with the class $\mathcal{C}$ of C^*-algebras. Form the class $\mathcal{K}$ as follows:
$$\mathcal{K} = \{(A, R_n^A) : A \in \mathcal{C}\}.$$
We suppress the k for readability; one checks that these are metric structures in an appropriate language.
- It suffices to see that $\mathcal{K}$ is an elementary class. Then by the corollary to the Beth definability theorem, R_n^k is a definable predicate in the language of C^*-algebras.
- This boils down to showing the $\mathcal{K}$ is closed under ultraproducts (think about ultraroots as an exercise).
- So fix some index set I and ultrafilter $\mathcal{U}$ on I. Suppose we have $(A_i, R_n^{A_i}) \in \mathcal{K}$ and let $A = \prod_{\mathcal{U}} A_i$.
- We want to show that $R_n^A = \lim_{i \to \mathcal{U}} R_n^{A_i}$.

A sketch of the proof, cont'd

- Now it is relatively straightforward to see that

$$R_n^A \leq \lim_{i \to \mathcal{U}} R_n^{A_i}.$$

- Suppose that the inequality is strict. Then one can find $\bar{a} \in A$, cpc maps φ and ψ as necessary such that

$$\|\bar{a} - \psi(\varphi(\bar{a}))\| < \lim_{i \to \mathcal{U}} R_n^{A_i}(\bar{a}_i).$$

- Consider ψ first. The image of the matrix units from $M_n(\mathbb{C})$ to A under a cpc map is a definable set. It follows then that there are cpc maps $\psi_i : M_n(\mathbb{C}) \to A_i$ such that $\psi = \lim_{\imath \to \mathcal{U}} \psi_i$.
- We now consider φ. This is subtler and we will cheat (but only a little) and let $n = 1$.
- So φ is a positive linear functional and we will consider $\varphi\restriction_{\bar{a}}$. We can find $\bar{a}_i \in A_i$ and φ_i on A_i such that

$$\varphi\restriction_{\bar{a}} = \lim_{i \to \mathcal{U}} \varphi_i\restriction_{\bar{a}_i}.$$

- Putting the φ_i's together with the ψ_i's is a contradiction to the strict inequality and we're done.

Model theoretic forcing and nuclearity

- This was somehow the whole point of the exercise: to obtain a model theoretic charactarization of nuclearity which would allow us to do forcing.
- This sometimes is called omitting types or the Baire category theorem. For me this is a Henkin construction.
- This would provide a new way to construct C^*-algebras in general and possibly new nuclear C^*-algebras.
- How does this work? Start with a theory T and countably many new constants symbols C.
- At each step we will have some finitely many open conditions $|\varphi(\bar{c}) - r| < \epsilon$ for some r, ϵ which are jointly consistent with T.
- We proceed inductively to both strengthen the conditions and also to impose further properties including possibly nuclearity.
- In the end, we will formally written down conditions on the constants and we produce a C^*-algebra from the closure of the implied algebra.

Crazy question/conjecture

Question

Suppose that A and B are two separable, unital, simple nuclear C^*-algebras with the same continuous theory and the same Elliott invariant. Then are A and B isomorphic?

All known counter-examples to the Elliott conjecture have distinct continuous theories.