Continuous Model Theory

Lecture 4: Ultrapowers of II_1 factors

Bradd Hart

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McDuff's question

- In her 1970 PLMS paper, McDuff proved that if M is a separable II_1 factor such that the central sequence algebra for M is non-commutative i.e. $M' \cap M^{\mathcal{U}}$ is not abelian for some (any) non-principal ultrafilter on $\mathbb{N}$, then $M \cong M \otimes \mathcal{R}$.
- She made systematic use of ultraproducts and the central sequence algebra in her work but noticed that it didn't seem to matter which non-principal ultrafilter she used.
- She asked if $\mathcal{U}$ and $\mathcal{V}$ were non-principal ultrafilters on $\mathbb{N}$, are $M' \cap M^{\mathcal{U}}$ and $M' \cap M^{\mathcal{V}}$ isomorphic?
- In fact, one could ask if $M^{\mathcal{U}}$ was isomorphic to $M^{\mathcal{V}}$.
- It turns out that both of these questions are model theory questions. Why?

Types

Fix a theory T in a language $\mathcal{L}$. We consider (partial) functions p on the space of formulas $\mathcal{F}_{\overline{x}}$ for a tuple $\bar{x}$ of sorted variables to $\mathbb{R}$.

Definition

1. p is a (partial) type if there is a model $\mathcal{M}$ of T and $\overline{a} \in \mathcal{M}$ of the appropriate sort such that $p(\varphi) = \varphi^{\mathcal{M}}(\overline{a})$ for all $\varphi \in \mathrm{dom}(p)$. We say that $\overline{a}$ realizes p.
2. p is called a complete type if the domain of p is $\mathcal{F}_{\overline{x}}$.

Fact

- *p is a type iff it is finitely satisfied i.e. if the restriction to every finite subset of its domain is a type.*
- *A complete type is a linear functional on $\mathcal{F}_{\overline{x}}$.*

A topology on the type space

We fix a language $\mathcal{L}$ and a complete theory T in this language. For a tuple of sorts $\overline{S}$ from $\mathcal{L}$, we define the set $S_{\overline{x}}(T)$ to be all complete types defined on $\mathcal{F}_{\overline{x}}$.
The logic topology on $S_{\overline{x}}(T)$ is the restriction of the weak-* topology on the dual space of $\mathcal{F}_{\overline{x}}$. Equivalently, the collection of sets

$$\{p \in S_{\overline{x}}(T) : p(\varphi) < r\} \text{ for every formula } \varphi \text{ and real number } r,$$

form the collection of basic open sets.

Fact

- *The logic topology on $S_{\overline{x}}(T)$ is compact and Hausdorff.*
- *If φ is a formula then the function f_φ from $S_{\overline{x}}(T)$ to $\mathbb{R}$ given by $p \mapsto p(\varphi)$ is continuous.*

A metric on the type space

Fix a complete theory T.

- Define a metric on $S_{\overline{x}}(T)$ as follows: for $p, q \in S_{\overline{x}}(T)$, $d(p, q)$ is the infinum of $d^{\mathcal{M}}(\overline{a}, \overline{b})$ where $\mathcal{M}$ ranges over all models of T, $\overline{a} \in \mathcal{M}$ is a realization of p and $\overline{b} \in \mathcal{M}$ is a realization of q. d is computed as the maximum of the values d_S as S ranges over the sorts in $\overline{S}$.
- Claim: d defines a metric on $S_{\overline{x}}(T)$.
- Notice that $d(p, q)$ is always realized - this follows by compactness as does the triangle inequality.

Proposition

The metric topology on $S_{\overline{x}}(T)$ refines the logic topology.

Saturation

Definition

We say a metric structure $\mathcal{M}$ is λ-saturated if whenever $A \subseteq M$ is of density character $< \lambda$ and p is a type over A then p is realized in $\mathcal{M}$. We say $\mathcal{M}$ is saturated if it is λ-saturated for λ the density character of $\mathcal{M}$.

Proposition

If $\mathcal{M}$ and $\mathcal{N}$ are saturated of the same theory and density character then $\mathcal{M} \cong \mathcal{N}$.

Proposition

If $\mathcal{M}$ is separable and $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$ then $\mathcal{M}^{\mathcal{U}}$ is $\aleph_1$-saturated i.e. any type over a separable subset of $\mathcal{M}^{\mathcal{U}}$ is realized in $\mathcal{M}^{\mathcal{U}}$.

Corollary

If CH holds, $\mathcal{M}$ is separable and $\mathcal{U}$ and $\mathcal{V}$ are two non-principal ultrafilters on $\mathbb{N}$ then $\mathcal{M}^{\mathcal{U}} \cong \mathcal{M}^{\mathcal{V}}$.

Set theoretic considerations

- One consequence of $\aleph_1$-saturation is that if $N \equiv M$ and N is separable then N embeds into $M^{\mathcal{U}}$ for any non-principal ultrafilter $\mathcal{U}$ - $M^{\mathcal{U}}$ is separably universal.
- Notice then that if we assume CH, McDuff's question has a positive answer. We can fix the separable model M and build an isomorphism of $M^{\mathcal{U}}$ and $M^{\mathcal{V}}$ which fixes M and hence the relative commutants would all be isomorphic as well.
- Assuming CH though doesn't actually get at the heart of this question.
- For this talk we will say that a property holds "necessarily" if it holds in all models of ZFC regardless of the value of the continuum.

Stability

Definition

A theory T is λ-stable if for any model $\mathcal{M}$ of T of density character λ, the type space over $\mathcal{M}$ with the metric topology has density character λ. A theory is stable if it is λ-stable for some λ.

Example

Infinite-dimensional Hilbert spaces: Every type over a Hilbert space with orthonormal basis *I* is determined by functions from *I* to $\mathbb{C}$. Dense among these are the types that are 0 at all but finitely many elements of *I*. So the theory of infinite-dimensional Hilbert spaces is λ-stable for all λ.

The order property

Definition

A theory T has the order property if there is a model $\mathcal{M}$ of T, a formula $\varphi(\bar{x}, \bar{y})$ and tuples from $\mathcal{M}$ $\bar{a}_1, \bar{a}_2, \ldots$ such that

$$\varphi(\bar{a}_i, \bar{a}_j) = 0 \text{ if } i \leq j \text{ and } 1 \text{ if } i > j.$$

Example

The Banach space c_0 of ω-sequences of real numbers which tend to 0. Let the formula $\varphi(x, y; u, v)$ be $2 - \|x + v\|$. If e_k is the sequence with 1 in the kth spot and 0 elsewhere and a_k is the sequence with 1 up to the kth spot and 0 afterwards then

$$\varphi(a_m, e_m; a_n, e_n) = 1 \text{ if } m < n \text{ and } 0 \text{ otherwise.}$$

Ultrapower characterization of stability

Theorem

For a separable complete theory T, the following are equivalent:

1. *T is stable.*
2. *If $\mathcal{M}$ is a separable model of T then for any non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, $\mathcal{M}^{\mathcal{U}}$ is necessarily saturated.*
3. *If $\mathcal{M}$ is a separable model of T then the isomorphism type of $\mathcal{M}^{\mathcal{U}}$ is necessarily unique for any non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$.*
4. *T does not have the order property.*

Sketch of a proof of the characterization

- The easiest of the implications is 2 implies 3: $\mathcal{M}^{\mathcal{U}}$ has density character continuum and so if all such are saturated then the isomorphism type is unique.
- 3 implies 4 is beyond the scope of this lecture but roughly, if one assumes that T has the order property and that CH fails then one is allowed to code cardinalities other than the continuum into an ultrapower of a separable model.
- 4 implies 1 is approximately the same as in the classical case (no pun intended). In the continuous case, if T is not stable then there will be a formula φ and some ϵ such that over some separable model M, there is a φ-type which ϵ-splits over every finite subset of M. One builds an order out of this.

Sketch of a proof of the characterization, cont'd

- 1 implies 2: stablity is used to develop a notion of independence known as forking which is axiomatically well-behaved. To see that $\mathcal{M}^{\mathcal{U}}$ is saturated when T is stable, one chooses a type p over an elementary submodel $\mathcal{N} \prec \mathcal{M}^{\mathcal{U}}$ of size less than the continuum. From stability, p does not fork over some separable $\mathcal{N}_0 \prec \mathcal{N}$. In $\mathcal{M}^{\mathcal{U}}$ it is possible to find continuum many independent realizations of $p|\mathcal{N}_0$ and since $\mathcal{N}$ has size less than the continuum, not all of these realizations can be dependent on $\mathcal{N}$. So one of them must realize p itself.

Proof that II_1 factors are unstable

- Consider the formula $\varphi(x, y; u, v) := \|[x, v]\|_2$.
- Let
$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$
Note
$$[a, b] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \|[a, b]\|_2 = 1.$$
- Thinking of $\mathcal{R}$ as $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \dots$ let
$$c_k = a \otimes a \otimes a \dots 1 \otimes 1 \otimes \dots \text{ (} k \text{ times)}$$
and
$$d_k = 1 \otimes 1 \dots b \otimes 1 \otimes 1 \dots \text{ in the k}^{th} \text{ spot.}$$

Proof that II_1 factors are unstable, cont'd

- So in $\mathcal{R}$, we have

$$\varphi(c_m, d_m; c_n, d_n) = 0 \text{ if } m < n \text{ and } 1 \text{ if } m \geq n.$$

- Since φ is quantifier-free and $\mathcal{R}$ can be embedded into any II_1 factor, all II_1 factors have the order property and hence are unstable.

The case of the relative commutant

- Fix a separable II_1 factor M. It is important to note that the theory of the relative commutant does not depend on the ultrafilter. After that, there are three cases.
- The first possibility is that $M' \cap M^{\mathcal{U}}$ is $\mathbb{C}$. In this case then it is $\mathbb{C}$ no matter what the ultrafilter. This is the “not property Γ” case.
- The second possibility is that $M' \cap M^{\mathcal{U}}$ is abelian. More about this in a minute.
- The third possibility is that $M' \cap M^{\mathcal{U}}$ is not abelian. One argues that it contains a copy of $M_2(\mathbb{C})$ and then uses model theory to show that it actually contains a copy of $\mathcal{R}$.
- Now since the formula φ which witnesses the order property is quantifier-free, one repeats the argument that we have non-isomorphic ultrapowers relativized to the relative commutant.

The case of the relative commutant: the abelian case

- Now back to the abelian case: $M' \cap M^{\mathcal{U}}$ is abelian.
- It is a tracial von Neumann algebra and since M is a II_1 factor, the relative commutant does not have a minimal projection.
- From the characterization of abelian von Neumann algebras, the relative commutant is isomorphic to $L^\infty(B)$ for the atomless probability algebra B of density character continuum given by its projections.
- By the work of Ben Ya'acov, Henson et al, the theory of atomless probability algebras is stable and has quantifier elimination.
- We can now repeat the argument that stable theories have unique ultrapowers relativized in this case to show that these probability algebras are all isomorphic independent of the ultrafilter.
- We conclude that the isomorphism type of $M' \cap M^{\mathcal{U}}$ is independent of $\mathcal{U}$.