McMaster pre-Putnam 2018

There are three questions and they are weighted equally; do as many problems as you can in the hour allotted. Your work must be justified to receive full marks.

1. Let x, y and z be positive real numbers.

   (a) If $x + y + z \geq 3$, is it necessarily true that
   $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq 3?$$

   (b) If $x + y + z \leq 3$, is it necessarily true that
   $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3?$$

2. Find all 6 digit integers n such that:

   (a) n is a perfect square and

   (b) the number formed by the last three digits of n is exactly one greater than the number formed by the first three digits of n. That is n might look like 123124, although this is not a square.

3. An increasing sequence of integers is said to be alternating if it starts with an odd term, the second term is even, the third term is odd, the fourth is even, and so on. The empty sequence (with no term at all!) is considered to be alternating. Let $A(n)$ denote the number of alternating sequences which only involve integers from the set $\{1, 2, ..., n\}$.

   (a) Show that $A(1) = 2$ and $A(2) = 3$.

   (b) Find the value of A(20), and prove that your value is correct.