McPutnam 2019

There are three questions and they are weighted equally; do as many problems as you can in the hour allotted. Your work must be justified to receive full marks.

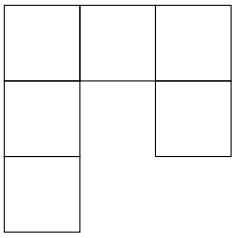
1. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that $\sum_{n=1}^{\infty} a_n$ diverges.

2. Consider the pair of four-digit positive integers

$$(M, N) = (3600, 2500).$$

   Notice that M and N are both perfect squares, with equal digits in two places, and differing digits in the remaining two places. Moreover, when the digits differ, the digit in M is exactly one greater than the corresponding digit in N. Find all pairs of four-digit positive integers (M, N) with these properties.

3. Define a hook to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure.



   Which m x n rectangles can be tiled by hooks?