Continuous Model Theory

Lecture 2: C^*-algebras and II_1 factors

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Łoś' Theorem

Theorem

Suppose $\mathcal{M}_i$ are $\mathcal{L}$-structures for all $i \in I, \mathcal{U}$ is an ultrafilter on I, $\varphi(\overline{x})$ is an $\mathcal{L}$-formula and $\overline{a} \in \mathcal{M} := \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ then

$$\varphi^{\mathcal{M}}(\overline{a}) = \lim_{i \to U} \varphi^{\mathcal{M}_i}(\overline{a}_i).$$

Satisfiability

Definition

- We say a set of sentences Σ in a language $\mathcal{L}$ is satisfied if there is an $\mathcal{L}$-structure $\mathcal{M}$ such that for every sentence in Σ holds in $\mathcal{M}$ i.e. for every $\varphi \in \Sigma$, $\varphi^{\mathcal{M}} = 0$.
- We say such a Σ is finitely satisfied if every finite subset of Σ is satisfied.
- For a set of sentence Σ and $\epsilon > 0$, the ϵ-approximation of Σ is

$$\{|\varphi| \dot{-} \epsilon : \varphi \in \Sigma\}$$

- Σ is approximately finitely satisfied if for every $\epsilon > 0$, the ϵ-approximation of Σ is finitely satisfiable.

Compactness

Theorem

TFAE for a set of sentences Σ in a language $\mathcal{L}$

- *Σ is satisfiable.*
- *Σ is finitely satisfiable.*
- *Σ is approximately finitely satisfiable.*

Embeddings and elementary submodels

- Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures such that the universe of $\mathcal{M}$ is a closed subset of $\mathcal{N}$. $\mathcal{M}$ is called a submodel if all functions and relations from $\mathcal{L}$ on $\mathcal{M}$ are the restriction of those from $\mathcal{N}$. We write $\mathcal{M} \subseteq \mathcal{N}$.
- For $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M}$ is an *elementary* submodel if, for every $\mathcal{L}$-formula $\varphi(\overline{x})$ and every $\overline{a} \in \mathcal{M}$, $\varphi^{\mathcal{M}}(\overline{a}) = \varphi^{\mathcal{N}}(\overline{a})$. We write $\mathcal{M} \prec \mathcal{N}$.
- An embedding between metric structures is a map which preserves the functions and relations. An embedding is elementary if its image is an elementary submodel of the range.
- For a theory T, Mod(T) is the category of models of T with elementary maps as morphisms. Such a class is called an elementary class.

Notice that by Łoś' Theorem, any metric structure $\mathcal{M}$ embeds elementarily into its ultrapower $\mathcal{M}^{\mathcal{U}}$ for any ultrafilter $\mathcal{U}$ via the diagonal embedding.

Downward Löwenheim-Skolem

Proposition (Tarski-Vaught)

If $\mathcal{M} \subseteq \mathcal{N}$ then $\mathcal{M}$ is an elementary submodel if for every formula $\varphi(x, \overline{y})$, $r \in \mathbb{R}$ and $\overline{a} \in \mathcal{M}$, if $(\inf_x \varphi(x, \overline{a}))^{\mathcal{N}} < r$ then there is $b \in \mathcal{M}$ such that $(\varphi(b, \overline{a}))^{\mathcal{N}} < r$.

Theorem (DLS)

Suppose that $\mathcal{N}$ is an $\mathcal{L}$-structure and A is a subset of $\mathcal{N}$. Then there is an elementary submodel $\mathcal{M} \subseteq \mathcal{N}$ such that

1. *A is contained in $\mathcal{M}$ and*
2. *for every sort S,*
$$\chi(S^{\mathcal{M}}) \leq \chi(A) + |\mathcal{L}|$$
where χ gives the density character of the given space.

Some abstract model theory

Theorem

For a class of $\mathcal{L}$-structures $\mathcal{C}$, TFAE

1. *$\mathcal{C}$ is an elementary class.*
2. *$\mathcal{C}$ is closed under isomorphisms, ultraproducts and elementary submodels.*
3. *$\mathcal{C}$ is closed under isomorphisms, ultraproducts and ultraroots.*

Theorem

Continuous first order logic is the maximal logic on metric structures which satisfies compactness, the downward Löwenheim-Skolem theorem and unions of elementary chains.

Linear operators

- Fix a Hilbert space H and consider a linear operator A on H. The operator norm of A is defined as

$$\|A\| := \sup\{\|Ax\| : x \in H, \|x\| = 1\}.$$

  If this is defined, we call A bounded.
- We write $B(H)$ for the algebra of all bounded operators on H.
- $B(H)$ carries a natural complex vector space structure and multiplication is composition. There is an adjoint operation defined via the inner product on H: for $A \in B(H)$, A^* satisfies, for all $x, y \in H$,

$$\langle Ax, y \rangle = \langle x, A^* y \rangle.$$

- The operator norm puts a normed linear structure on $B(H)$ and the norm satisfies the C^*-identity $\|A^*A\| = \|A\|^2$ for all $A \in B(H)$.

C^*-algebras

Definition

- A *concrete* C^*-algebra is a norm closed *-subalgebra of $B(H)$ for some Hilbert space H.
- An *abstract* C^*-algebra is Banach *-algebra which satisfies the C^*-identity.

Example

- For any Hilbert space H, $B(H)$ is a concrete C^*-algebra. In particular, $M_n(\mathbb{C})$, $n \times n$ complex matrices, is a C^*-algebra for all n.
- $C(X)$, all continuous functions on a compact, Hausdorff space X is an abelian (abstract) C^*-algebra. The norm is the sup-norm. By a result of Gelfand and Naimark, these are all the unital abelian C^*-algebras.

C^*-algebras

Example

- C^*-algebras are closed (as abstract C^*-algebras) under direct sums and direct limits with *-homomorphism embeddings as connecting maps.
- Any finite-dimensional C^*-algebra is the direct sum of finitely many copies of matrix algebras.

Theorem (Gelfand-Naimark-Segal)

Every abstract C^*-algebra is isomorphic to a concrete C^*-algebra.

C^*-algebraic ultraproducts

- A dead give-away that model theory is involved is that operator algebraists are using ultraproducts.
- Suppose A_i are C^*-algebras for all $i \in I$ and that $\mathcal{U}$ is an ultrafilter on I. Consider the bounded product

$$\prod^b A_i := \{\bar{a} \in \prod A_i : \lim_{i \to \mathcal{U}} \|a_i\| < \infty\}$$

  and the two-sided ideal $c_{\mathcal{U}}$

$$\{\bar{a} \in \prod^b A_i : \lim_{i \to \mathcal{U}} \|a_i\| = 0\}.$$

  The ultraproduct, $\prod_{\mathcal{U}} A_i$ is defined as $\prod^b A_i / c_{\mathcal{U}}$.
- One checks that multiplication, addition and the adjoint are well-defined coordinatewise modulo $c_{\mathcal{U}}$.

C^*-algebras as metric structures

- We treat C^*-algebras as we did Banach spaces: there are sorts for each ball of radius $n \in \mathbb{N}$.
- There are inclusion maps between the balls. Additionally there are functions for the restriction of all the operations to the balls. This involves the addition, multiplication, scalar multiplication and the adjoint.
- The metric is given via the operator norm as $\|x - y\|$ on each ball.
- It is routine to check that all of these functions are uniformly continuous (the only issue is multiplication and this holds because we have restricted the norm).
- The sorts are complete since C^*-algebras are complete.
- Do we have an elementary class? You would think so since C^*-algebras are closed under ultraproducts and subalgebras.

Axioms for C^*-algebras

- There are many universal axioms expressing that a C^*-algebra is a Banach *-algebra. These involve saying that two terms are equal or that some norm or other is equal to or less than something else.
- For instance, to say $\tau(\bar{x}) = \sigma(\bar{x})$ we need to write $\sup_{\bar{x}} d(\tau(\bar{x}), \sigma(\bar{x}))$ which is awful so we write the first and mean the second.
- For the metric, we have $d(x, 0) = \|x\|$ and $d(x, y) = \|x - y\|$. One can now write out the axioms for a Banach space.
- Include $\|x^*\| = \|x\|$ and the C^*-identity, $\|x^* x\| = \|x\|^2$.
- Now comes the fussy bits about using balls: we have

$$\sup_{x \in B_1} \|x\| \leq 1 \text{ and } \sup_{x \in B_n} \min\{1 \dot{-} \|x\|, \inf_{y \in B_1} \|x - y\|\}.$$

- This feels a little awkward since operator algebraists know that C^*-algebras are closed under subalgebras and we know that should mean the axioms are universal. They are if you introduce enough terms!

A second topology

- The weak operator topology on $B(H)$ is induced by the family of semi-norms given by, for every $\zeta, \eta \in H$,

$$A \mapsto |\langle A\zeta, \eta\rangle|.$$

- $M \subseteq B(H)$ is a von Neumann algebra if it is a unital *-algebra closed in the weak operator topology.
- Equivalently, any unital *-algebra $M \subseteq B(H)$ which satisfies $M'' = M$ is a von Neumann algebra where

$$M' = \{A \in B(H) : [A, B] = 0 \text{ for all } B \in M\}.$$

Traces and tracial von Neumann algebras

Definition

A linear functional τ on a C*-algebra M is a (finite, normalized) **trace** if

- it is positive ($\tau(a^*a) \geq 0$ for all $a \in M$),
- $\tau(a^*a) = \tau(aa^*)$ for all $a \in M$, and
- $\tau(1) = 1$.

We say it is faithful if $\tau(a^*a) = 0$ implies $a = 0$.

A tracial von Neumann algebra M is a von Neumann algebra with a faithful trace τ. τ induces a norm on M

$$\|a\|_2 = \sqrt{\tau(a^*a)}.$$

Examples

- $M_n(\mathbb{C})$ with the normalized trace is a tracial vNa; $B(H)$ for infinite-dimensional H is not.
- Inductive limits of tracial von Neumann algebras are tracial von Neumann algebras. In particular, $\mathcal{R}$, **the** inductive limit of the $M_n(\mathbb{C})$'s is a tracial von Neumann algebra called the hyperfinite II_1 factor.
- $L(G)$ - suppose H has an orthonormal generating set ζ_h for $h \in G$. Let u_g for $g \in G$ be the operator determined by

  $$u_g(\zeta_h) = \zeta_{gh}.$$

  $L(G)$ is the von Neumann algebra generated by the u_g's. It is tracial: for $a \in L(G)$, let $\tau(a) = \langle a(\zeta_e), \zeta_e \rangle$.

Tracial ultraproducts

Suppose M_i are von Neumann algebras with faithful traces τ_i for all $i \in I$ and $\mathcal{U}$ is an ultrafilter on I. Again, the bounded product is

$$\prod^b M_i := \{\bar{a} \in \prod M_i : \lim_{i \to \mathcal{U}} \|a_i\| < \infty\}$$

and now consider the two-sided ideal

$$c_{\mathcal{U}} = \{\bar{a} \in \prod^b M_i : \lim_{i \to \mathcal{U}} \tau_i(a_i^* a_i) = 0\}.$$

The ultraproduct, $\prod_{\mathcal{U}} M_i$, is defined as $\prod^b M_i / c_{\mathcal{U}}$. It is a tracial von Neumann algebra with the trace given by $\tau(\bar{x}) = \lim_{i \to \mathcal{U}} \tau_i(x_i)$.

Tracial vNas as metric structures

- The tracial ultraproducts actually guide us in figuring out how to see tracial von Neumann algebras as metric structures.
- As with C*-algebras, we introduce sorts for the balls of operator norm n for each $n \in \mathbb{N}$. The big difference is that the operator norm will **NOT** be in the language.
- The basic functions are again considered as partitioned across the sorts together with the necessary inclusion maps. We also have the trace which formally we have to break into its real and imaginary parts.
- The metric on each ball is induced by the 2-norm; it is complete on each ball. It is critical that the 2-norm is restricted to a bounded ball.

Do tracial vNas form an elementary class?

- Let $\mathcal{K}$ be the class of metric structures arising from tracial von Neumann algebras as on the previous slide. Is $\mathcal{K}$ an elementary class? Let's check this semantically.
- Tracial ultraproducts of von Neumann algebras are equivalent to the ultraproduct in the metric structure sense for tracial von Neumann algebras viewed as metric structures. So $\mathcal{K}$ is closed under ultraproducts (and isomorphism).
- The closure under subalgebras is handled by the very same trick as with C^*-algebras.
- That $\mathcal{K}$ and the category of tracial von Neumann algebras is an equivalence of categories is taken care of by the Kaplansky density theorem.

II_1 factors

- A von Neumann algebra whose centre is $\mathbb{C}$ is called a factor.
- A tracial factor is type I if all its projections have rational trace and is type II_1 if the range of the trace on projections is [0,1].
- $\mathcal{R}$, $\mathcal{R}^{\mathcal{U}}$, $\prod_U M_n(\mathbb{C})$ and $L(F_n)$ are all II_1 factors.
- The class of II_1 factors is an elementary class.

Property Γ

- Consider M any II_1 factor and the partial type

$$p(x) = \{\|[x, m]\|_2 : m \in M\}.$$

- (Murray-von Neumann) M has property Γ if p is not algebraic relative to the theory of M. That is, p has a realization outside of M in some elementary extension of M. Property Γ is elementary.
- $\mathcal{R}$ has property Γ; $\prod_{\mathcal{U}} M_n(\mathbb{C})$ does not have property Γ; neither does $L(F_n)$.

Central sequence algebras

- Suppose that M is a separable II_1 factor; $M \prec M^{\mathcal{U}}$ and consider all realizations of p in $M^{\mathcal{U}}$ - it is $M' \cap M^{\mathcal{U}}$, the relative commutant or the central sequence algebra. It is also a tracial von Neumann algebra.
- There are three cases (McDuff):
  - M does not have property Γ,
  - M has property Γ and the relative commutant is abelian (and does not depend on $\mathcal{U}$), or
  - M has a non-abelian relative commutant (it is type II_1).
- McDuff asked if in the third case, the isomorphism type depends on $\mathcal{U}$.

Theorem (Farah, H., Sherman)

The answer to McDuff's question is yes because the theory of all II_1 factors is unstable!

Theories of II_1 factors

Theorem (Boutonnet, Chifan, Ioana)

There are continuum many theories of II_1 factors. In fact, McDuff's original examples of continuum many non-isomorphic II_1 factors are not elementarily equivalent.

- The free group factor problem asks if $L(F_m)$ and $L(F_n)$ are not isomorphic for $m \neq n$. A model theoretic version of this question: are $L(F_m)$ and $L(F_n)$ elementarily equivalent?
- We do know that $L(F_\infty)$ and $\prod_{\mathcal{U}} L(F_n)$ have the same $\forall\exists$-theory for non-principal $\mathcal{U}$.
- Related questions: for non-principal $\mathcal{U}$, is the theory of $\prod_{\mathcal{U}} M_n(\mathbb{C})$ independent of $\mathcal{U}$? How are the theory of ultraproducts of matrix algebras related to the theories of free group factors?