Continuous Model Theory

Lecture 3: Practical definability

Bradd Hart

May 9, 2017

Tracial vNas as metric structures

- The tracial ultraproducts actually guide us in figuring out how to see tracial von Neumann algebras as metric structures.
- As with C*-algebras, we introduce sorts for the balls of operator norm n for each $n \in \mathbb{N}$. The big difference is that the operator norm will **NOT** be in the language.
- The basic functions are again considered as partitioned across the sorts together with the necessary inclusion maps. We also have the trace which formally we have to break into its real and imaginary parts.
- The metric on each ball is induced by the 2-norm; it is complete on each ball. It is critical that the 2-norm is restricted to a bounded ball.

Do tracial vNas form an elementary class?

- Let $\mathcal{K}$ be the class of metric structures arising from tracial von Neumann algebras as on the previous slide. Is $\mathcal{K}$ an elementary class? Let's check this semantically.
- Tracial ultraproducts of von Neumann algebras are equivalent to the ultraproduct in the metric structure sense for tracial von Neumann algebras viewed as metric structures. So $\mathcal{K}$ is closed under ultraproducts (and isomorphism).
- The closure under subalgebras is handled by the very same trick as with C^*-algebras.
- That $\mathcal{K}$ and the category of tracial von Neumann algebras is an equivalence of categories is taken care of by the Kaplansky density theorem.

II_1 factors

- A von Neumann algebra whose centre is $\mathbb{C}$ is called a factor.
- A tracial factor is type I if all its projections have rational trace and is type II_1 if the range of the trace on projections is [0,1].
- $\mathcal{R}$, $\mathcal{R}^{\mathcal{U}}$, $\prod_U M_n(\mathbb{C})$ and $L(F_n)$ are all II_1 factors.
- The class of II_1 factors is an elementary class.

Property Γ

- Consider M any II_1 factor and the partial type

$$p(x) = \{\|[x, m]\|_2 : m \in M\}.$$

- (Murray-von Neumann) M has property Γ if p is not algebraic relative to the theory of M. That is, p has a realization outside of M in some elementary extension of M. Property Γ is elementary.
- $\mathcal{R}$ has property Γ; $\prod_{\mathcal{U}} M_n(\mathbb{C})$ does not have property Γ; neither does $L(F_n)$.

Central sequence algebras

- Suppose that M is a separable II_1 factor; $M \prec M^{\mathcal{U}}$ and consider all realizations of p in $M^{\mathcal{U}}$ - it is $M' \cap M^{\mathcal{U}}$, the relative commutant or the central sequence algebra. It is also a tracial von Neumann algebra.
- There are three cases (McDuff):
  - M does not have property Γ,
  - M has property Γ and the relative commutant is abelian (and does not depend on $\mathcal{U}$), or
  - M has a non-abelian relative commutant (it is type II_1).
- McDuff asked if in the third case, the isomorphism type depends on $\mathcal{U}$.

Theorem (Farah, H., Sherman)

The answer to McDuff's question is yes because the theory of all II_1 factors is unstable!

Theories of II_1 factors

Theorem (Boutonnet, Chifan, Ioana)

There are continuum many theories of II_1 factors. In fact, McDuff's original examples of continuum many non-isomorphic II_1 factors are not elementarily equivalent.

- The free group factor problem asks if $L(F_m)$ and $L(F_n)$ are not isomorphic for $m \neq n$. A model theoretic version of this question: are $L(F_m)$ and $L(F_n)$ elementarily equivalent?
- We do know that $L(F_\infty)$ and $\prod_{\mathcal{U}} L(F_n)$ have the same $\forall\exists$-theory for non-principal $\mathcal{U}$.
- Related questions: for non-principal $\mathcal{U}$, is the theory of $\prod_{\mathcal{U}} M_n(\mathbb{C})$ independent of $\mathcal{U}$? How are the theory of ultraproducts of matrix algebras related to the theories of free group factors?

Types

Fix a theory T in a language $\mathcal{L}$. We consider (partial) functions p on the space of formulas $\mathcal{F}_{\overline{x}}$ for a tuple $\bar{x}$ of sorted variables to $\mathbb{R}$.

Definition

1. p is a (partial) type if there is a model $\mathcal{M}$ of T and $\overline{a} \in \mathcal{M}$ of the appropriate sort such that $p(\varphi) = \varphi^{\mathcal{M}}(\overline{a})$ for all $\varphi \in \mathrm{dom}(p)$. We say that $\overline{a}$ realizes p.
2. p is called a complete type if the domain of p is $\mathcal{F}_{\overline{x}}$.

Fact

- *p is a type iff it is finitely satisfied i.e. if the restriction to every finite subset of its domain is a type.*
- *A complete type is a linear functional on $\mathcal{F}_{\overline{x}}$.*

A topology on the type space

We fix a language $\mathcal{L}$ and a complete theory T in this language. For a tuple of sorts $\overline{S}$ from $\mathcal{L}$, we define the set $S_{\overline{x}}(T)$ to be all complete types defined on $\mathcal{F}_{\overline{x}}$.
The logic topology on $S_{\overline{x}}(T)$ is the restriction of the weak-* topology on the dual space of $\mathcal{F}_{\overline{x}}$. Equivalently, the collection of sets

$$\{p \in S_{\overline{x}}(T) : p(\varphi) < r\} \text{ for every formula } \varphi \text{ and real number } r,$$

form the collection of basic open sets.

Fact

- *The logic topology on $S_{\overline{x}}(T)$ is compact and Hausdorff.*
- *If φ is a formula then the function f_φ from $S_{\overline{x}}(T)$ to $\mathbb{R}$ given by $p \mapsto p(\varphi)$ is continuous.*

What is a formula?

Relative to a theory T, define a pseudo-norm on $\mathcal{F}_{\bar{x}}$ by

$$\|\varphi(\bar{x})\| := \sup\{|\varphi^{\mathcal{M}}(\bar{a})| : \bar{a} \in \mathcal{M}\}.$$

This puts a normed linear structure on $\mathcal{F}_{\bar{x}}$.

Proposition

The following are equivalent:

1. *f is a continuous function from $S_{\overline{x}}(T)$ to $\mathbb{R}$.*
2. *f is the uniform limit of functions of the form f_φ i.e. for every n there is a formula φ_n such that for all p, $|f(p) - p(\varphi_n)| \leq 1/n$.*

Definition

A Cauchy sequence of formulas $\overline{\varphi}$ in $\mathcal{F}_{\overline{x}}$ will be called a definable predicate and interpreted in an $\mathcal{L}$-structure $\mathcal{M}$ by

$$\overline{\varphi}^{\mathcal{M}}(\overline{a}) = \lim_{n\to\infty} \varphi_n^{\mathcal{M}}(\overline{a}).$$

Of course what we are doing is extending the notion of formula to the Banach space generated by $\mathcal{F}_{\overline{x}}$.

Zero sets and distance predicates

Fix a theory T in a language $\mathcal{L}$ and a model $\mathcal{M}$ of T.

- For a definable predicate $\varphi(\bar{x})$, the zero set of φ in $\mathcal{M}$ is

$$Z(\varphi^{\mathcal{M}}) := \{\bar{a} \in \mathcal{M} : \varphi^{\mathcal{M}}(\bar{a}) = 0\}.$$

- If X is a non-empty closed subset of some product of sorts in $\mathcal{M}$ we call $P(x) = d(x, X) = \inf\{d(x, y) : y \in X\}$ the distance predicate for X.

Definable sets

We introduce the category Met where the objects are bounded metric spaces and the morphisms are isometries.

Definition

Suppose we have a theory T in a language $\mathcal{L}$ and S_i for $i \leq n$ are sorts in $\mathcal{L}$. We call a functor

$$X : \mathrm{Mod}(T) \to \mathrm{Met}$$

a uniform assignment relative to T if for every model $\mathcal{M}$ of T, $X(\mathcal{M})$ is a closed subset of $\prod_{j=1}^{m} S_j^{\mathcal{M}}$ and X is just restriction on morphisms. This assignment is called a *definable set* if, for all formulas $\psi(\bar{x}, \bar{y})$, the functions defined for all $\mathcal{M}$, models of T, by

$$\sup_{\bar{x} \in X(\mathcal{M})} \psi^{\mathcal{M}}(\bar{x}, \bar{y}) \quad \text{and} \quad \inf_{\bar{x} \in X(\mathcal{M})} \psi^{\mathcal{M}}(\bar{x}, \bar{y})$$

are definable predicates for T.

Examples

- Sorts are definable sets as are products of sorts.
- Products of definable sets are definable.
- The range of a term is definable.
- For example, in the operator algebra setting, the set of self-adjoint elements in any algebra is definable as is the set of positive elements: $(x + x^*)/2$ and x^*x are the two terms in question.

Critical remarks about definable sets

- A natural source of uniform assignments is the zero-set of any definable predicate.
- If an assignment is a definable set then it is the assignment arising from the zero-set of some definable predicate. Just choose $\psi(\bar{x}, \bar{y}) := d(\bar{x}, \bar{y})$ and parse $\inf_{\bar{x} \in X(\mathcal{M})} \psi(\bar{x}, \bar{y})$.
- The definition of definable set could be read

  "Definable sets are those sets you can quantify over."

  Notice in the discrete case, you can quantify over the solution set of any formula.
- There are lots of zero sets which are NOT definable sets; we will see some in a few slides.

A useful lemma

Lemma (MTFMS, 2.10)

Suppose that $F, G : X \to [0, 1]$ *are functions such that*

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in X\ (F(x) \leq \delta \implies G(x) \leq \epsilon)$$

Then there exists an increasing, continuous function $\alpha : [0, 1] \to [0, 1]$ *such that* $\alpha(0) = 0$ *and*

$$\forall x \in X\ (G(x) \leq \alpha(F(x))).$$

A second characterization of definable sets

Theorem

Suppose that X is a uniform assignment relative to a theory T. Then the following are equivalent:

1. *This assignment is a definable set.*
2. *The distance predicate $d(\bar{x}, X)$ is a definable predicate for T.*
3. *There is a definable predicate $\varphi(\bar{x})$ so that for any model of T, $\mathcal{M}$, $Z(\varphi^{\mathcal{M}}) \subseteq X(\mathcal{M})$ and for every $\epsilon > 0$ there is a $\delta > 0$,*

$$\text{if } \varphi^{\mathcal{M}}(\bar{a}) < \delta \text{ then } d(\bar{a}, X(\mathcal{M})) \leq \epsilon.$$

A proof

For (2) implies (1), fix a formula ψ. It is uniformly continuous so using MTFMS 2.10, we can find continuous α such that for all $\bar{x}, \bar{y}$ and $\bar{z}$

$$|\psi(\bar{x}, \bar{z}) - \psi(\bar{y}, \bar{z})| \leq \alpha(d(\bar{x}, \bar{y})).$$

Consider

$$\inf_{\bar{z}}(\psi(\bar{x}, \bar{z}) + \alpha(d(\bar{z}, X))) \text{ and } \inf_{\bar{z} \in X} \psi(\bar{x}, \bar{z}).$$

The claim is that these are equal and the first is a definable predicate.

Examples

- Definable sets are closed under unions.
- In the operator algebra setting, the set of projections in a given algebra is a definable set.
- This is related to the notion of weakly stable relations: a definable predicate $\varphi(\bar{x})$ is said to be weakly stable (or be a weakly stable relation) relative to the theory T if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\mathcal{M}$ is a model of T and $\varphi^{\mathcal{M}}(\bar{a}) < \delta$ for some $\bar{a} \in \mathcal{M}$ then there is $\bar{b} \in \mathcal{M}$ such that $\varphi(\bar{b}) = 0$ and $d(\bar{a}, \bar{b}) \leq \epsilon$.
- The zero set of a weakly stable relation is clearly a definable set.
- There are many examples of weakly stable relations in the operator algebraic literature; for instance, that an n^2-tuple x_{ij} for $i, j \leq n$ are the matrix units of a unital copy of $M_n(\mathbb{C})$.

A third characterization of definable sets

Theorem

Suppose that X is a uniform assignment relative to a theory T. Then the following are equivalent:

1. *This assignment is a definable set.*
2. *For all sets I, ultrafilters $\mathcal{U}$ on I and models of T, $\mathcal{M}_i$ for $i \in I$, if $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$ then*

$$X(\mathcal{M}) = \prod_{\mathcal{U}} X(\mathcal{M}_i).$$

Examples of definable sets

- The set of normal elements in $B(H)$ is not definable (or more correctly the set of normal elements is not a definable set relative to the theory of $B(H)$ or the theory of C^*-algebras.)
- The ball of radius 1 around a point in the ball of radius 1 in a Hilbert space. Far more generally, if the underlying metric space has geodesics then balls will be definable sets.
- Ultrametrics give examples that are not definable. Here is a toy example: on the interval $[0, 2]$ define the metric d

$$d(x, y) = \begin{cases} \max\{x, y\} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

  If we fix 0 as a constant than the zero-set of $d(0, x) \leq 1$ doesn't survive ultrapowers. These types of examples arise naturally from metrics associated to certain valuations.

Beth definability

Theorem

Suppose that $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2$ are languages, $\mathcal{L}_0$ and $\mathcal{L}_1$ have the same sorts and T is an $\mathcal{L}_2$-theory. Further suppose whenever $\mathcal{M}$ is a model of T, $\mathcal{M}_0 = \mathcal{M} \restriction_{\mathcal{L}_0}$ and $\mathcal{U}$ is an ultrafilter then:

if $f : \mathcal{M}_0 \to \mathcal{M}_0^{\mathcal{U}}$ is an $\mathcal{L}_0$-elementary map then f is also an $\mathcal{L}_1$-map.

Then every $\mathcal{L}_1$-formula is T-equivalent to a definable predicate in $\mathcal{L}_0$.

Corollary

Suppose $\mathcal{L}_0 \subseteq \mathcal{L}_1$ are two languages with the same sorts, T_i is a theory in $\mathcal{L}_i$ for $i = 0, 1$ and the forgetful functor

$$F : Mod(T_1) \to Mod(T_0)$$

given by restriction to $\mathcal{L}_0$ is an equivalence of categories. Then every $\mathcal{L}_1$-formula is T_1-equivalent to a definable predicate in $\mathcal{L}_0$.

Nuclear algebras

- A linear map $\varphi : A \to B$ is positive if $\varphi(a^*a) \geq 0$ for all $a \in A$ (positive elements go to positive elements).
- φ is completely positive if for all n, $\varphi^{(n)} : M_n(A) \to M_n(B)$ is positive.
- φ is contractive if $\|\varphi\| \leq 1$; *-homomorphisms are cpc maps.

Definition (Completely positive approximation property)

A C*-algebra A is nuclear if for every $\bar{a} \in A$ and $\epsilon > 0$ there is an n and cpc maps

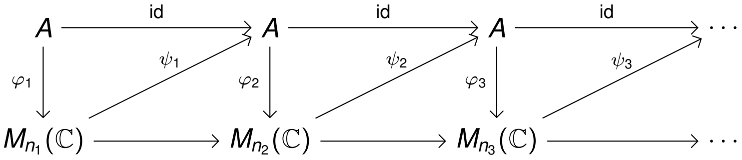
$$\varphi : A \to M_n(\mathbb{C}) \text{ and } \psi : M_n(\mathbb{C}) \to A$$

such that

$$\|\bar{a} - \psi\varphi(\bar{a}))\| < \epsilon.$$

- Examples: Abelian C*-algebras, $M_n(\mathbb{C})$
- Inductive limits of nuclear algebras; nuclear algebras are closed under $\otimes$ and direct sum.

A helpful picture



The general classification problem is to give a complete (usable) set of invariants for all (unital), separable, simple nuclear algebras.

Model theoretic characterization of nuclear algebras

Consider, for $k, n \in \mathbb{N}$, the predicate defined on A_1^k by

$$R_n^k(\bar{a}) = \inf_{\varphi,\psi} \|\bar{a} - \psi(\varphi(\bar{a}))\|$$

where $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ range over cpc maps.

Theorem (FHLRTVW)

R_n^k is a definable predicate in the language of C^*-algebras.

Corollary

A C^*-algebra is nuclear if it satisfies, for all k,

$$\sup_{\bar{x}} \inf_n R_n^k(\bar{x}).$$

A sketch of the proof

- Fix n and k and start with the class $\mathcal{C}$ of C^*-algebras. Form the class $\mathcal{K}$ as follows:

$$\mathcal{K} = \{(A, R_n^A) : A \in \mathcal{C}\}.$$

  We suppress the k for readability; one checks that these are metric structures in an appropriate language.
- It suffices to see that $\mathcal{K}$ is an elementary class. Then by the corollary to the Beth definability theorem, R_n^k is a definable predicate in the language of C^*-algebras.
- This boils down to showing the $\mathcal{K}$ is closed under ultraproducts (think about ultraroots as an exercise).
- So fix some index set I and ultrafilter $\mathcal{U}$ on I. Suppose we have $(A_i, R_n^{A_i}) \in \mathcal{K}$ and let $A = \prod_{\mathcal{U}} A_i$.
- We want to show that $R_n^A = \lim_{i \to \mathcal{U}} R_n^{A_i}$.

A sketch of the proof, cont'd

- Now it is relatively straightforward to see that

$$R_n^A \leq \lim_{i \to \mathcal{U}} R_n^{A_i}.$$

- Suppose that the inequality is strict. Then one can find $\bar{a} \in A$, cpc maps φ and ψ as necessary such that

$$\|\bar{a} - \psi(\varphi(\bar{a}))\| < \lim_{i \to \mathcal{U}} R_n^{A_i}(\bar{a}_i).$$

- Consider ψ first. The image of the matrix units from $M_n(\mathbb{C})$ to A under a cpc map is a definable set. It follows then that there are cpc maps $\psi_i : M_n(\mathbb{C}) \to A_i$ such that $\psi = \lim_{\imath \to \mathcal{U}} \psi_i$.
- We now consider φ. This is subtler and we will cheat (but only a little) and let $n = 1$.
- So φ is a positive linear functional and we will consider $\varphi\restriction_{\bar{a}}$. We can find $\bar{a}_i \in A_i$ and φ_i on A_i such that

$$\varphi\restriction_{\bar{a}} = \lim_{i \to \mathcal{U}} \varphi_i\restriction_{\bar{a}_i} .$$

- Putting the φ_i's together with the ψ_i's is a contradiction to the strict inequality and we're done.

Crazy question/conjecture

Question

Suppose that A and B are two separable, unital, simple nuclear C^*-algebras with the same continuous theory and the same Elliott invariant. Then are A and B isomorphic?

All known counter-examples to the Elliott conjecture have distinct continuous theories.