Practical Definability

Bradd Hart
McMaster University

July 25, 2017

Case study: Connes and his ultraproduct

- In his “Injective factors” paper, Connes gives several characterizations of the condition known as property Γ (Theorem 2.1).
- He uses a technical proposition (Proposition 1.3.1) in which he gives a variant of the ultraproduct construction for II_1 factors acting on a Hilbert space.
- As we continuous model theorists know, it is difficult to handle II_1 factors acting on Hilbert spaces and Connes recites the reasons for this even in that paper.
- Still, he uses an ultraproduct so as model theorists we are duty bound to figure out what is going on.

Continuous model theory in a few slides

- Metric language: multi-sorted with designated metric symbol for each sort; function and relation symbols as in classical first order logic but with specified uniform continuity moduli. Relation symbols also come with a bound - this applies to the metric symbols as well.
- Formulas: Atomic formulas are built as usual. Connectives are all real-valued functions; quantifiers are sup and inf.
- Metric structures: Each sort is interpreted as a complete metric space. Functions and relations are interpreted as uniformly continuous functions as specified. Relations are real-valued and respect their bound. Formulas are interpreted naturally.

Ultraproducts of metric structures

Fix a language $\mathcal{L}$, an index set I, an ultrafilter $\mathcal{U}$ on I and $\mathcal{L}$-structures $\mathcal{M}_i$ for $i \in I$.

Definition

The ultraproduct of the $\mathcal{M}_i$'s with respect to $\mathcal{U}$, $\prod_\mathcal{U} \mathcal{M}_i$ is the $\mathcal{L}$-structure $\mathcal{M}$ defined as follows:

1. for every sort S, $S^\mathcal{M} = \prod_\mathcal{U} S^{\mathcal{M}_i}$ with metric $d_S^\mathcal{M} = \lim_{i \to \mathcal{U}} d_S^{\mathcal{M}_i}$,
2. for every function symbol f with range S

$$f^\mathcal{M}(\bar{x}) = \langle f^{\mathcal{M}_i}(\bar{x}_i) : i \in I \rangle / d_S^\mathcal{M}, \text{ and}$$

3. for every relation symbol R,

$$R^\mathcal{M} = \lim_{i \to \mathcal{U}} R^{\mathcal{M}_i}.$$

If all of the $\mathcal{M}_i$'s are a fixed $\mathcal{N}$, we call this the ultrapower and write $\mathcal{N}^\mathcal{U}$.

Łoś' Theorem

Theorem

Suppose $\mathcal{M}_i$ are $\mathcal{L}$-structures for all $i \in I, \mathcal{U}$ is an ultrafilter on I, $\varphi(\overline{x})$ is an $\mathcal{L}$-formula and $\overline{a} \in \mathcal{M} := \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ then

$$\varphi^{\mathcal{M}}(\overline{a}) = \lim_{i \to U} \varphi^{\mathcal{M}_i}(\overline{a}_i).$$

Definable sets

Met is the category of bounded complete metric spaces with isometries as morphisms.

Definition

Suppose we have a theory T in a language $\mathcal{L}$ and S_i for $i \leq n$ are sorts in $\mathcal{L}$. We call a functor

$$X : \mathrm{Mod}(T) \to \mathrm{Met}$$

a uniform assignment relative to T if for every model $\mathcal{M}$ of T, $X(\mathcal{M})$ is a closed subset of $\prod_{j=1}^{m} S_j^{\mathcal{M}}$ and on morphisms, X is just restriction. This assignment is called a *definable set* if, for all formulas $\psi(\bar{x}, \bar{y})$, the functions defined for all $\mathcal{M}$, models of T, by

$$\sup_{\bar{x} \in X(\mathcal{M})} \psi^{\mathcal{M}}(\bar{x}, \bar{y}) \quad \text{and} \quad \inf_{\bar{x} \in X(\mathcal{M})} \psi^{\mathcal{M}}(\bar{x}, \bar{y})$$

are equivalent to formulas in T.

Semantic approach to definability

Theorem

Suppose that X is a uniform assignment relative to a theory T. Then the following are equivalent:

1. *This assignment is a definable set.*
2. *For all sets I, ultrafilters $\mathcal{U}$ on I and models of T, $\mathcal{M}_i$ for $i \in I$, if $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$ then*

$$X(\mathcal{M}) = \prod_{\mathcal{U}} X(\mathcal{M}_i).$$

II_1 factors in one slide

- $B(H)$ is the set of all bounded operators on a Hilbert space H. It has the natural structure of a complex *-algebra.
- The weak operator topology is generated by the following open sets: for $A \in B(H)$, $\epsilon > 0$ and $x, y \in H$,

$$\{B \in B(H) : |\langle (A - B)x, y\rangle| < \epsilon\}.$$

- A von Neumann algebra (concretely) is a unital *-subalgebra of $B(H)$ which is closed in the weak operator topology. It is a factor if its centre is $\mathbb{C}$.
- A II_1 factor is an infinite-dimensional factor with a faithful normal trace i.e. a suitably continuous positive linear functional τ which satisfies $\tau(1) = 1$, $\tau(ab) = \tau(ba)$ for all a and b and $\tau(a^*a) = 0$ iff $a = 0$.
- The class of II_1 factors for an elementary class in continuous model theory.

Standard form

- If $\mathcal{M}$ is a II_1 factor with faithful, normal trace τ, let $L^2(M, \tau)$ be the Hilbert space determined by putting the inner product $\langle x, y\rangle = \tau(y^* x)$ on $\mathcal{M}$. $\mathcal{M}$ acts on $L^2(\mathcal{M}, \tau)$ on the left and 1 is a cyclic separating vector.
- In fact in the II_1 factor case, this is the standard form of such a representation:
  - A II_1 factor $\mathcal{M}$ acting on a Hilbert space H with a cyclic, separating vector Ω,
  - the inner product on H being determined by $\langle a\Omega, \Omega\rangle = \tau(a)$, and
  - a conjugate linear isometric involution J on H determined by $J(a\Omega) = a^*\Omega$. Note

$$JMJ = M' := \{a \in B(H) : ab = ba \text{ for all } b \in M\}.$$

- By results of Araki, Connes and Haagerup, the standard form is essentially unique.

Standard form is imaginary

- $L^2(\mathcal{M}, \tau)$ lives among the imaginaries of $\mathcal{M}$. This requires a little work; here are a few words:
  - Elements in the unit ball of the Hilbert space $L^2(\mathcal{M}, \tau)$ can appear inside the operator norm ball of arbitrarily high norm.
  - One needs to consider the countable product of balls of operator norm n for all n in order to see Cauchy sequences wrt the 2-norm.
  - If you are careful, you can quotient this product in order to obtain the unit ball of the given Hilbert space.
- As a bonus, we get that the commutant, $\mathcal{M}'$, on $L^2(M, \tau)$ lives as an imaginary definable set in $\mathcal{M}$.
- Connes ultraproduct construction is the observation that the standard form of the ultraproduct of II_1 factors is obtained from the imaginaries in the ultraproduct. In particular, $L^2(\mathcal{M}^{\mathcal{U}}, \tau) \neq L^2(M, \tau)^{\mathcal{U}}$!