# John von Neumann, model theorist

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# Quantum physicist?

- Did he say John von Neumann, model theorist?
- Did he mean John von Neumann, quantum physicist?
- This would seem more likely and is actually related to today's talk.
- JvN believed that a good model of quantum mechanics would be certain collections of linear operators acting on an infinite-dimensional Hilbert space. I am not going to dwell on this too much because ...
- I don't completely understand this and ...
- I really do mean JvN, model theorist. Nevertheless let's explore one operator algebra related to quantum mechanics.

# Let's introduce $R$ and some terminology

- $M_n(\mathbb{C})$ of course acts on $\mathbb{C}^n$ as this is all linear operators on this Hilbert space.
- $M_n(\mathbb{C}) \hookrightarrow M_{nk}(\mathbb{C}) \cong M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ or more descriptively, by the map
$$A \mapsto \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}$$
- $$M_2 \hookrightarrow M_4 \hookrightarrow M_8 \hookrightarrow \ldots$$
  and this acts on
$$\mathbb{C}^2 \hookrightarrow \mathbb{C}^4 \hookrightarrow \mathbb{C}^8 \hookrightarrow \ldots$$
- The inductive limit, call it $M_\infty$, acts on the limit of the Hilbert spaces which we will call $H$.

# $R$, cont'd

- $M_\infty$ is an algebra acting on an infinite-dimensional Hilbert space. What algebraic operations does it have?
- $+, \cdot, *$, scalar multiplication by $\lambda \in \mathbb{C}$ and there are two relevant norms:
- The first is the usual operator norm:

$$\|A\| = \sup\{\|Ax\| : x \in H, \|x\| \leq 1\}$$

- We also have a trace on this algebra (Warning: this is a normalized trace so for $A \in M_n(\mathbb{C}), tr(A) = \frac{\sum a_{ii}}{n}$).
- Notice that both the operator norm, the trace and all the algebraic operations are preserved by the embeddings that go into making $M_\infty$.

# $R$, cont'd

- The trace has several important properties:
  - it is faithful ($tr(a^*a) = 0$ implies $a = 0$),
  - normal ($tr(1) = 1$),
  - positive ($tr(a^*a) \geq 0$ for all $a$),
  - it is a linear functional and
  - for all $a, b$ it satisfies $tr(ab) = tr(ba)$
- Von Neumann regarded the existence of a trace on the collection of relevant operators as essential. Notice that $B(H)$ does not support a trace for infinite-dimensional $H$.
- One thing that is clear is that the trace provides an inner product on the algebra: $\langle x, y \rangle = tr(y^*x)$ and we write $\|A\|_2 = \sqrt{tr(A^*A)}$. This provides a second norm on $M_\infty$.
- An important aside: For $e_{11} \in M_n$, $\|e_{11}\| = 1$ but $\|e_{11}\|_2 = \dfrac{1}{\sqrt{n}}$.

# Finally $R$

- In analysis, one always takes the closure of unions so what we really want is the closure of $M_\infty$ in what topology?
- The weak-* topology: on $B(H)$, a basic open set about $A$ looks like, for $\zeta, \eta \in H$ and $\epsilon > 0$,

$$O^{\epsilon}_{\zeta,\eta} = \{B \in B(H) : |\langle B\zeta, \eta\rangle - \langle A\zeta, \eta\rangle| < \epsilon\}$$

- $R$, the hyperfinite $\mathrm{II}_1$ factor (centre is $\mathbb{C}$), is the weak-* closure of $M_\infty$ in $B(H)$.
- $R$ was von Neumann's original model. It is in fact the weak-* closure of any inductive limit of matrix algebras.

# Operator algebraist?

- OK, so it looks like the talk should have been titled "John von Neumann, operator algebraist".
- This would be reasonable. He wrote a series of papers with F.J.Murray beginning in the mid-30's called "On rings of operators" which is where the study of operator algebras began.
- One thing that they did in these papers was introduce what we now called von Neumann algebras. A *-subalgebra of $B(H)$ is called a von Neumann algebra if it is closed in the weak-* topology.
- They classified the building block von Neumann algebras, the factors, into types I, II and III.
- The only factors that support a trace were the type $\mathrm{I}_n$ factors, $M_n(\mathbb{C})$, and the $\mathrm{II}_1$ factors of which $R$ is an example.
- In fact, if there is a trace on a factor then it is unique.

# Model theorist

- JvN was particularly interested in $\mathrm{II}_1$ factors and to what extent $R$ was canonical or categorical.
- In 1942, in Portugaliae Mathematica, von Neumann published the paper "Approximative properties of matrices of high finite order".
- Here is an excerpt from the introduction:

> Our interest will be concentrated in this note on the conditions in ... $M_n$ when $n$ is *finite*, but very great. This is an approach to the study of the infinite dimensional which differs essentially from the usual one. The usual approach consists in studying an actually infinite dimensional ... Hilbert space. We wish to investigate instead the *asymptotic* behaviour of ... $M_n$ for finite $n$ when $n \to \infty$.

# Model theorist

- Now this is starting to look like a model theoretic question; he is essentially asking whether the generic model for a class of algebras (the class of matrix algebras) is the same as the model (or theory) obtained asymptotically.
- Some times it is: if we look at the class of finite graphs, the generic model is the random graph and, by the 0-1 law for finite graphs, the asymptotic theory is also that of the random graph.
- Some times it is not: if we look at the class of finite fields, the generic model(s) are algebraically closed (one for every prime) but the asymptotic theory, by Ax, is that of pseudo-finite fields.

# Main results

**Theorem (Version 1, JvN, 9.3 PM)**

*There exists* $\epsilon > 0$ *such that for every* $n$ *and* $k > 1$*, there is* $A \in M_{nk}$ *with* $\|A\| \leq 1$ *such that for all* $B \in M_n$ *and all unitary* $U \in M_{nk}$, $\|A - U^* B U\|_2 \geq \epsilon$.

**Theorem (Version 2, JvN, 23.1 PM)**

*Given* $\delta > 0$ *there is* $\epsilon > 0$ *such that for every* $n$ *there is* $A \in M_n$ *with* $\|A\| \leq 1$ *such that for every* $B \in M_n$ *with* $\|B\| \leq 1$ *and* $\|B^* B - B B^*\|_2 \leq \epsilon$ *and* $\|AB - BA\|_2 \leq \epsilon$ *we have* $\|B - tr(B) \cdot 1\|_2 \leq \delta$.

- This last formulation begins to look like a logical statement but in what logic?

# A crash course in continuous model theory

- Continuous logic is a new logic developed over the past decade by Ben Ya'acov, Henson and others (it had precursors in the work of Henson, Keisler and others).
- The underlying structures are complete, bounded metric spaces and functions and relations on these are uniformly continuous.
- Relations take values in some pre-assigned closed bounded interval in the reals (note the metric itself is a relation in this regard being bounded).
- Formal syntax is very similar to classical logic - terms are obtained by composition (with the added twist that a uniform continuity modulus is inherited from the function symbols - the uniform continuity mentioned above is part of the language!)

# Crash course, cont'd

- Basic atomic formulas are obtained by substituting terms into relations.
- All continuous functions are connectives i.e. if $\varphi_1, \ldots, \varphi_n$ are formulas and $f : \mathbb{R}^n \to \mathbb{R}$ then $f(\varphi_1, \ldots, \varphi_n)$ is a formula.
- By the restriction on the range of relations, these formulas will be interpreted as bounded, uniformly continuous functions on a metric structure.
- sup and inf are the two quantifiers in this logic i.e. if $\varphi$ is a formula then both $\sup_x \varphi$ and $\inf_x \varphi$ are formulas.
- Again, since formulas will have bounded ranges when interpreted in a metric structure, inductively the interpretation of sup and inf will make sense.

# $\mathrm{II}_1$ factors as metric structures

- Suppose that $A \subseteq B(H)$ is a $\mathrm{II}_1$ factor i.e. an infinite-dimensional von Neumann algebra that is a factor and has a trace.
- Consider the structure where the underlying metric space is the operator norm unit ball $A_1 = \{x \in A : \|x\| \leq 1\}$ with the metric given by the 2-norm, $d(x, y) = \|x - y\|_2$.
- The functions in the language are all *-polynomials which map the operator norm unit ball of *any* $\mathrm{II}_1$ factor back into itself. For instance, $\frac{x+y}{2}, xy, x^*, \lambda x$ for $|\lambda| \leq 1$ etc.

**Theorem (Farah,H.,Sherman)**

*The theory of operator norm unit balls of $II_1$ factors is an elementary class.*

# Ultraproducts of metric structures

- Suppose that $M_i$ for $i \in I$ are metric structures for the same signature $L$ and $U$ is an ultrafilter on $I$.
- Define functions coordinatewise on $M = \prod_{i \in I} M_i$ and relations as follows: for $R \in L$,

$$R^M(\bar{m}^1, \ldots, \bar{m}^n) = \lim_{i \to U} R^{M_i}(m_i^1, \ldots, m_i^n)$$

- Note that this includes the relation $d$ on $M$ which might be a pseudo-metric after this construction.
- $\prod_{i \in I} M_i/U$, the ultraproduct, is the quotient of $M$ by the pseudo-metric induced by $d$.

# Ultraproducts of metric structures, cont'd

**Theorem (Łoś Theorem)**

*If $M_i$ for $i \in I$ are metric structures for the same signature L, U is an ultrafilter on I, $M = \prod_{i \in I} M_i / U$ and $\varphi$ is an L-formula then*

$$\varphi^M(\bar{m}) = \lim_{i \to U} \varphi^{M_i}(\bar{m}_i)$$

- This implies that if $U$ is an ultrafilter on $I$ and $M$ is a metric structure then the diagonal embedding of $M$ into $M^U$ is elementary.

# Property $\Gamma$

- Von Neumann and Murray answered the question about $R$'s categoricity after JvN published his article in Port. Math. They introduced something called property $\Gamma$.
- If $A$ is a $II_1$ factor and $U$ is a non-principal ultrafilter on $\mathbb{N}$, the relative commutant of $A$ in $A^U$, written $A' \cap A^U$ is

$$\{B \in A^U : B \text{ commutes with all } C \in A\}$$

- We say that $A$ has property $\Gamma$ if $A' \cap A^U \neq \mathbb{C}$.
- Having property $\Gamma$ is independent of the choice of ultrafilter. In fact, property $\Gamma$ is an elementary property.

# Property $\Gamma$, cont'd

- $R$ has property $\Gamma$. In fact, McDuff showed that for a separable $\mathrm{II}_1$ factor $A$ either
  - $A$ does not have property $\Gamma$,
  - $A$ has property $\Gamma$, $A' \cap A^U$ is abelian and determined up to isomorphism by $A$, or
  - $A$ has property $\Gamma$ and $A' \cap A^U$ has type $\mathrm{II}_1$. She asked if the isomorphism type here was unique.

**Theorem (Farah, H., Sherman)**

*A tracial von Neumann algebra is stable iff it has type 1.*

**Corollary**

*A strong no to McDuff's question.*

# Back to von Neumann

- The asymptotic theory of matrix algebras would be the theory of ultraproducts of matrix algebras.
- It is not hard to show that for a non-principal ultrafilter $U$ on $\mathbb{N}$, $\prod_{n\in\mathbb{N}} M_n(\mathbb{C})/U$ is a $\mathrm{II}_1$ factor.
- Following von Neumann, Murray and McDuff's lead, we ask if such an algebra has property $\Gamma$ - this is an elementary question about the continuous theory of the ultraproduct!
- The answer is no and the proof follows from von Neumann's calculations in his 1942 paper - the calculation is contained in my third paper with Farah and Sherman.
- Is this the end of the story ... ?

# The theory of $R$

$R$ has an interesting property - it embeds into any $\mathrm{II}_1$ factor and any embedding of it into any model of its theory is elementary which implies that $R$ is the prime model of its theory.

**Theorem (Goldbring,H., Sinclair)**

*$Th(R)$ does not have quantifier elimination. In fact, the theory of tracial von Neumann algebras does not have a model companion.*

**Theorem (GHS)**

*$Th(R)$ is probably not model complete.*

**Theorem (Farah, Goldbring, H.)**

*$Th(R)$ might be $\forall\exists$-axiomatizable but no other theory of $II_1$ factors satisfying $Th_\forall(R)$ is.*

# Theories of matrix ultraproducts

- The best question here is whether any two matricial ultraproducts via non-principal ultrafilters on $\mathbb{N}$ have the same theory.
- The current state of knowledge is pathetic - there is either one such theory or continuum many.
- They all have the same $(1+\epsilon)$-theory i.e. they agree on sentences with a single sup quantifier over self-adjoint elements and an arbitrary inf quantifier.
- We don't know if they have the same $\forall\exists$-theory; note that property $\Gamma$ is $\forall\exists$.
- We do know that the theory of any such matrix ultraproduct (unique or not) is *not* $\forall\exists$-axiomatizable.