Model theory and operator algebras: Status report and open problems

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Joint work with a cast of thousands

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Outline

- Metric structures and continuous model theory
- Primer on operator algebras for model theorists
- Von Neumann algebras and II_1 factors
- Issues of decidability
- Nuclear algebras and the Elliott classification problem
- Nuclearity as omitting types

Continuous logic; the logic of metric structures

- A metric structure consists of three types of objects:
  - $\mathcal{S}$, a collection of bounded, complete metric spaces called sorts,
  - $\mathcal{F}$, a collection of uniformly continuous functions on these sorts, and
  - $\mathcal{R}$, a collection of bounded, uniformly continuous functions on the sorts into $\mathbb{R}$.

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- A continuous language $\mathcal{L}$ encodes this information:
  - There are sorts and sorted variables together with a distinguished symbol d_S for the metric in each sort,
  - Function symbols together with uniform continuity moduli, and
  - Relation symbols together with uniform continuity moduli and a bound.

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  - There are sorts and sorted variables together with a distinguished symbol d_S for the metric in each sort,
  - Function symbols together with uniform continuity moduli, and
  - Relation symbols together with uniform continuity moduli and a bound.
- Atomic formulas are built exactly as in discrete first order logic.
- Connectives: If $f : \mathbb{R}^n \to \mathbb{R}$ and $\varphi_1, \ldots, \varphi_n$ are formulas then $f(\varphi_1, \ldots, \varphi_n)$ is a formula.
- Quantifiers: If φ is a formula then $\sup_x \varphi$ and $\inf_x \varphi$ are both formulas.

Continuous model theory

- Formulas are interpreted in metric structures ($\mathcal{L}$-structures) as expected. Formulas take values in $\mathbb{R}$, bounded independent of the $\mathcal{L}$-structure.
- Ultraproducts: For a collection $(X_i d_i)$, $i \in I$, of uniformly bounded metric spaces and an ultrafilter U on I, let $X = \prod_I X_i$ and

$$d(\bar{x}, \bar{y}) = \lim_{i \to U} d_i(x_i, y_i)$$

  This is a pseudo-metric on X and we call X/d the metric ultraproduct of the X_i's.
- Ultraproducts of L-structures are obtained by taking the metric ultraproduct sort by sort, interpreting functions coordinatewise and defining relations via ultralimits.

Continuous model theory, cont'd

- Łoś Theorem: If M_i, $i \in I$ are $\mathcal{L}$-structures, U is an ultrafilter on I and $M = \prod_U M_i$ then for any formula φ

$$\varphi^M(\bar{m}) = \lim_{i \to U} \varphi^{M_i}\varphi(\bar{m}_i)$$

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- Corollary: The compactness theorem holds for continuous logic.
- If M and N are $\mathcal{L}$-structures and M is a substructure of N the we say $M \prec N$ if for all formulas and all $\bar{m} \in M$, $\varphi^M(\bar{m}) = \varphi^N(\bar{m})$.
- Downward Lowenheim-Skolem: If $\mathcal{L}$ is a countable language and N is an $\mathcal{L}$-structure then there is a separable M such that $M \prec N$.

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- Downward Lowenheim-Skolem: If $\mathcal{L}$ is a countable language and N is an $\mathcal{L}$-structure then there is a separable M such that $M \prec N$.
- There is a Lindstrom Theorem for continuous logic so this is the correct logic for metric structures if you want basic model theory properties like compactness, DLS, unions of elementary chains, etc.

Operator algebra basics

- Fix a Hilbert space H and let $B(H)$ be all bounded linear operators on H; for $A \in B(H)$, $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$.
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- A C*-algebra $M \subseteq B(H)$ is a complex *-algebra which is closed in the norm topology.
- Examples: $M_n(C), B(H)$
- Finite dimensional C*-algebras are direct sums of $M_n(C)$'s - C*-algebras are closed under direct sum.
- C*-algebras are closed under inductive limits: the inductive limits of $M_n(C)$'s are the separable UHF algebras; inductive limits of finite-dimensional algebras are the AF algebras.

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- C*-algebras are closed under inductive limits: the inductive limits of $M_n(C)$'s are the separable UHF algebras; inductive limits of finite-dimensional algebras are the AF algebras.
- (GNS) There is an abstract characterization of C*-algebras: They are Banach *-algebras satisfying the C*-identity, $\|a^*a\| = \|a\|^2$.

A second topology

- The weak *-topology on $B(H)$ is induced by the family of semi-norms give, for every $\zeta, \eta \in H$,

$$A \mapsto |\langle A\zeta, \eta\rangle|$$

- $M \subseteq B(H)$ is a von Neumann algebra if it is a unital *-algebra closed in the weak *-topology.
- Equivalently, any unital *-algebra $M \subseteq B(H)$ which satisfies $M'' = M$ is a von Neumann algebra;
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- Traces: A linear functional τ on a C*-algebra M is a trace if it is positive ($\tau(a^*a) \geq 0$ for all $a \in M$), $\tau(a^*a) = \tau(aa^*)$ for all $a \in M$ and $\tau(1) = 1$. We say it is faithful if $\tau(a^*a) = 0$ implies $a = 0$.

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- Examples: $M_n(C)$ with the normalized trace; not $B(H)$
- Direct sums of tracial von Neumann algebras
- Inductive limits of tracial von Neumann algebras. In particular, $\mathcal{R}$, the hyperfinite II_1 factor is the inductive limit of the $M_n(C)$'s.

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- Inductive limits of tracial von Neumann algebras. In particular, $\mathcal{R}$, the hyperfinite II_1 factor is the inductive limit of the $M_n(C)$'s.
- $L(F_n)$ - suppose H has an orthonormal generating set ζ_h for $h \in F_n$. Let u_g for $g \in F_n$ be the operator determined by

$$u_g(\zeta_h) = \zeta_{gh}$$

  $L(F_n)$ is the von Neumann algebra generated by the u_g's. It is tracial: for $a \in L(F_n)$, let $\tau(a) = \langle a(\zeta_e), \zeta_e \rangle$.

Operator algebras as metric structures

- For a C*-algebra or a tracial von Neumann algebra M, consider sorts S_n for each $n \in N$, for the ball of operator norm n in M.
- Functions like $+, \cdot, *$ and scalar multiplication are broken up across these sorts - there are also inclusion maps to keep everything straight.
- The metrics: in the case of C*-algebras, the metric on each ball is just the one determined by the operator norm; in the case of tracial von Neumann algebra, the metric is induced by the 2-norm.

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Theorem (Farah-H.-Sherman)

- *The class of C*-algebras forms an elementary class.*
- *The class of tracial von Neumann algebras forms an elementary class*

Some consequences of this model theory

- The standard construction of the ultraproduct of C*-algebras is the same as taking the ultraproduct as metric structures.
- Tracial ultraproducts of von Neumann algebras, introduced by McDuff, are also equivalent to the ultraproduct in the metric structure sense for tracial von Neumann algebras.
- $\prod_U M_n(C)$ is a C*-algebra; $\prod_U M_n(C)$ is also a tracial von Neumann algebra albeit with a different metric.
- A von Neumann algebra whose centre is C is called a factor - this can be expressed as a sentence in continuous logic. To say that a tracial von Neumann algebra is type II_1 just means that it has a projection with irrational trace which can also be expressed in continuous logic.
- $\mathcal{R}$ is a II_1 factor and so is $\mathcal{R}^{\mathcal{U}}$; $\prod_U M_n(C)$ is also a II_1 factor; $L(F_n)$ is also a II_1 factor.

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- (JvN) M has property Γ if p is not algebraic. Property Γ is elementary by its definition.
- $\prod_U M_n(C)$ does not have property Γ; neither does $L(F_n)$.
- Consider $M \prec M^U$ and all realizations of p in M^U - it is $M' \cap M^U$, the relative commutant - it is also a von Neumann algebra.

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  - M has a non-abelian relative commutant (it is type II_1).
- McDuff asked if in the third case, the isomorphism type depends on U. We (Farah, H., Sherman) answered yes because the theory of II_1 factors is unstable!

Consequences

- We know three distinct elementary classes of II_1 factors
  - the theories of $\prod_U M_n(C)$, $L(F_n)$,
  - classical examples with property Γ and abelian relative commutant (Dixmier-Lance), and
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  - classical examples with property Γ and abelian relative commutant (Dixmier-Lance), and
  - the theory of $\mathcal{R}$.
- Questions:
  - Are all II_1 factors without property Γ elementarily equivalent?
  - Does the theory of $\prod_U M_n(C)$ depend on U?
  - Is $L(F_n) \equiv \prod_U M_n(C)$?
  - Is there a role for free probability to answer any of these questions?

The theory of $\mathcal{R}$

- $\mathcal{R}$ is the atomic model of its theory; any embedding of it into any other model of its theory is automatically elementary.
- $Th(\mathcal{R})$ is not model complete; in particular, it does not have quantifier elimination (FGHS;GHS).
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- What does this mean for a continuous theory? Is there an algorithm such that given a sentence φ and $\epsilon > 0$, we can compute $\varphi^{\mathcal{R}}$ to within ϵ.
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- Do we know such an axiomatization? No!
- We do have a recursive axiomatization of all tracial von Neumann algebras - this is a universal class so what do we know about $Th_\forall(\mathcal{R})$? Is it decidable?

A little background

- If A is any separable II_1 tracial von Neumann algebra then $\mathcal{R} \hookrightarrow A$;
- If $A \equiv_\forall \mathcal{R}$ then $A \hookrightarrow \mathcal{R}^U$.
- Equivalently, if $A \hookrightarrow \mathcal{R}^U$ then $Th_\forall(A) = Th_\forall(\mathcal{R})$.
- So if all separable II_1 tracial von Neumann algebras embed into $\mathcal{R}^U$ then $Th_\forall(\mathcal{R})$ is decidable.

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- To me, this says that this problem is very hard or that $Th(\mathcal{R})$ is undecidable (or both).

Nuclear algebras

- A linear map $\varphi : A \to B$ is positive if $\varphi(a^* a) \geq 0$ for all $a \in A$ (positive elements go to positive elements).
- φ is completely positive if for all n, $\varphi^{(n)} : M_n(A) \to M_n(B)$ is positive.
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Definition

A C*-algebra A is nuclear if for every $\bar{a} \in A$ and $\epsilon > 0$ there is an n and cpc maps $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ such that $\|\bar{a} - \psi\varphi(\bar{a}))\| < \epsilon$.

- Examples: Abelian C*-algebras, $M_n(C)$
- Inductive limits of nuclear algebras; nuclear algebras are closed under $\otimes$ and direct sum so AF and UHF algebras are nuclear.

The classification programme for separable nuclear algebras

The general problem

The general classification problem is to give a good classification scheme for all (unital), separable, simple nuclear algebras.

- Elliott classified all separable AF algebras and provided a template for classifying many more classes of nuclear algebras.
- This programme isn't arbitrary or crazy - see Winter's diagram.

The Elliott conjecture

Any separable, unital, simple nuclear algebra is determined, up to isomorphism, by its Elliott invariant.

The Elliott invariant

- Consider the equivalence relation $\sim$ on projections in A given by $p \sim q$ iff there is some $v \in A$, $vpv^* = q$ and $v^*qv = p$.
- The *-homomorphism $\Phi_n : M_n(A) \to M_{n+1}(A)$ defined by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

  and let $M_\infty = \lim_n M_n(A)$.
- Let $V(A) = Proj(M_\infty(A))/\sim$.
- $V(A)$ has an additive structure defined as follows: if $p, q \in V(A)$ then $p \oplus q$ is $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$
- $K_0(A)$ is the Grothendieck group generated from $(V(A), \oplus)$ and $K_0^+(A)$ is the image of $V(A)$ in $K_0(A)$; if A is unital then the constant $[1_A]$ corresponds to the identity in A.
- The Elliott invariant is $Ell(A) = ((K_0(A), K_0^+(A), [1_A]), K_1(A), Tr(A), \rho_A)$ where:
- $K_1(A) = K_0(C_0((0,1), A))$, $Tr(A)$ is the set of traces on A and ρ_A is the natural pairing of $Tr(A)$ and $K_0(A)$.

Why might model theory be involved?

Toms counter-examples

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Question/conjecture

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Question/conjecture

Separable, unital, simple C*-algebras are determined by their Elliott invariant and their theory.

- The isomorphism problem for AF algebras is not smooth in terms of Borel equivalence.
- The continuous theory of a metric structure is a smooth invariant.
- Conclusion (CCFGHMSS): there must be two elementarily equivalent AF algebras which are not isomorphic - name two!

How might model theory be involved?

- We know how to build models in ways different from operator algebraists: Henkin constructions and Fraïssé classes.
- They build up from the bottom via algebraic operations to form bootstrap classes.
- A test case for Henkin constructions is whether one can capture the notion of nuclearity via a Henkin construction.

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Theorem (FHLRTVW)

In the language of C-algebras, there are countably many partial types such that a C*-algebra omits these types iff it is nuclear.*

Sketch of a proof

- Fix $k, n \in N$ and define a relation $R_n(\bar{a})$ for $\bar{a} \in A_1^k$ by

$$\inf \varphi, \psi \|\bar{a} - \psi\varphi(\bar{a})\|$$

  where $\varphi : A \to M_n(C)$ and $\psi : M_n(C) \to A$ are cpc maps.
- It is possible to prove that if $A = \prod_U A_i$ then

$$R_n^A = \lim_{i \to U} R_n^{A_i}$$

- This means that the class of structures (A, R_n^A) is a conservative extenstion of the class C*-algebras.
- Hence, by Beth definability, R_n is equivalent to a formula in the language of C*-algebras.
- Omit the types, $p_m(\bar{x}) = \{R_n(\bar{x}) \geq 1/m : n \in N\}$ for all k-tuples $\bar{x}$.