Continuous Model Theory

Lecture 4: Urysohn space and C*-algebras

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June 14, 2016

Urysohn space

- The simplest continuous language is one which has only one sort and one relation symbol - the metric symbol. There are a lot of interesting metric spaces. Let's restrict ourselves to one's with diameter 1 i.e. where the metric is bounded by 1.
- Here is a construction of a universal, separable metric space - first constructed by Urysohn.
- We first record some facts about finite metric spaces:
  - (Amalgamation) If $f : A \to B$ and $g : A \to C$ are isometries and all spaces are finite then there is a finite D and isometries $h : B \to D$ and $j : C \to D$ such that $hf = jg$.
  - Since a 1-point space embeds into any metric space, we also have joint embedding.
- Consider a finite metric space A and the set of one-point extensions $S(A)$; we topologize $S(A)$ with the help of the space of 1-Lipschitz maps on A, $L(A)$.
- We put a metric on $L(A)$:

$$d(f, g) = \max_x |f(x) - g(x)|$$

$L(A)$ is a compact metric space.

Urysohn space, cont'd

- Identify $a \in A$ with $f_a : X \to [0,1]$ by $f_a(x) = d(x,a)$. One checks that $a \mapsto f_a$ is an isometry.
- If $B = A \cup \{b\}$, a 1-point extension of A, then identify b with $f_b(x) = d(x,b)$ for $x \in X$. One checks that $f_b \in L(A)$ and this is an isometric embedding over A.
- The key point is that the space of 1-point extensions of A is separable and compact.
- Now we build a metric space $\mathcal{U}$ as the completion of the union of an increasing chain of finite spaces X_n.
- For each finite $F \subseteq X_n$ we keep track of a countable dense subset of 1-point extensions of F and promise to amalgamate them all eventually.
- The inductive stage then looks like: $F \subseteq X_n$ and G, a 1-point extension of F. We let X_{n+1} be an amalgamation of X_n and G over F.

Urysohn space, cont'd

Theorem

- *The metric space $\mathcal{U}$ is separable, complete and universal i.e. isometrically embeds all separable metric spaces.*
- *$\mathcal{U}$ is ultrahomogeneous.*
- *The theory of $\mathcal{U}$ is separably categorical.*

Urysohn space, cont'd

- Instead of a full sketch of a proof, let's identify some axioms that hold in $\mathcal{U}$ and one cute construction. I will leave the rest as an exercise.
- If $\{a_1, \ldots, a_n\}$ is an enumeration of an n-element metric space A then let $\mathrm{Con}_A(x_1, \ldots, x_n)$, the configuration formula for A, be

  $$\max\{|d(x_i, x_j) - r_{ij}| : i, j \leq n\}$$

  where $r_{ij} = d(a_i, a_j)$. Clearly, if $\mathrm{Con}_A(a_1, \ldots, a_n) = 0$ and $\mathrm{Con}_A(b_1, \ldots, b_n) = 0$ then the map $a_i \mapsto b_i$ is an isometry.
- What if $\mathrm{Con}_A(b_1, \ldots, b_n) < \varepsilon$? Is there some finite metric space containing B and an isometric copy of A not too far away? Notice the connection to definable sets.
- Yes there is and let's construct it for definiteness.

Urysohn space, cont'd

- We have some space $B = \{b_1, \ldots, b_n\}$ such that $\mathrm{Con}_A(b_1, \ldots, b_n) = \Delta$.
- Build a metric space with elements $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ with the given metrics on A and B.
- Set $d(a_i, b_i) = \Delta/2$ for all $i \leq n$ and for $i, j \leq n$

$$d(a_i, b_j) = \min\{d(a_i, a_k) + d(b_k, b_j) + \Delta/2 : k \leq n\}.$$

- One checks that this defines a metric on these $2n$ points and the copy of A lies $\Delta/2$ away from the copy of B.
- Now suppose that we have a one point extension of A, $C = A \cup \{c\}$. Freely amalgamate B and C over A.
- I am interested in the value of $\mathrm{Con}_C(b_1, \ldots, b_n, c)$. The interesting cases are $d(c, b_i)$ for $i \leq n$. But

$$|d(c, b_i) - d(c, a_i)| \leq d(a_i, b_i) = \Delta/2$$

  so $\mathrm{Con}_C(b_1, \ldots, b_n, c) = \Delta$.
- What did we just achieve?

Urysohn space, cont'd

- Let A be any finite metric space and C, a one point extension. Consider the sentence

$$\varphi_{A,C} := \sup_{x_1,\ldots,x_n} |\mathrm{Con}_A(x_1,\ldots,x_n) - \inf_y \mathrm{Con}_C(x_1,\ldots,x_n,y)|.$$

- I claim that these sentences hold in $\mathcal{U}$. Since $\mathcal{U}$ is ultrahomogeneous and embeds all finite metric spaces, if one fixes $B = \{b_1,\ldots,b_n\}$ in $\mathcal{U}$ with $\mathrm{Con}_A(b_1,\ldots,b_n) = \Delta$ then by the construction on the previous slide, we can find c such that $\mathrm{Con}_C(b_1,\ldots,b_n,c) = \Delta$.
- These sentences allow one to do an approximate back and forth argument showing that $Th(\mathcal{U})$ is separably categorical.

Open question about Urysohn space

- The construction of Urysohn space feels like a Fraïssé construction and is in a technical sense. This makes it a generic object for the class of finite metric spaces.
- Is it also in some sense a random object? More precisely:
- Is Urysohn space elementarily equivalent to an ultraproduct of finite metric spaces?

Linear operators

- Fix a Hilbert space H and consider an linear operator A on H. The operator norm of A is defined as

$$\|A\| := sup\{\frac{\|Ax\|}{\|x\|} : x \in H\}$$

  if this is defined and then we call A bounded.
- We write $B(H)$ for the algebra of all bounded operators on H.
- $B(H)$ carries a natural complex vector space structure and multiplication is composition. There is an adjoint operation defined via the inner product on H: for $A \in B(H)$, A^* satisfies, for all $x, y \in H$,

$$\langle Ax, y \rangle = \langle x, A^* y \rangle.$$

- The operator norm puts a normed linear structure on $B(H)$ and the norm satisfies the C*-identity $\|A^* A\| = \|A\|^2$ for all $A \in B(H)$.

C*-algebras

Definition

- A *concrete* C*-algebra is a norm closed *-subalgebra of $B(H)$ for some Hilbert space H.
- An *abstract* C*-algebra is Banach *-algebra which satisfies the C*-identity.

Example

- For any Hilbert space H, $B(H)$ is a concrete C*-algebra. In particular, $M_n(\mathbb{C})$, $n \times n$ complex matrices, is a C*-algebra for all n.
- $C(X)$, all continuous functions on a compact, Hausdorff space X is an abelian (abstract) C*-algebra. The norm is the sup-norm. By a result of Gelfand and Naimark, these are all the unital abelian C*-algebras.

C*-algebras

Example

- C*-algebras are closed (as abstract C*-algebras) under direct sums and direct limits with *-homomorphism embeddings as connecting maps.
- Any finite-dimensional C*-algebra is the direct sum of finitely many copies of matrix algebras.

Theorem (Gelfand-Naimark-Sigal)

Every abstract C^*-algebra is isomorphic to a concrete C^*-algebra.

C*-algebraic ultraproducts

- A dead give-away that model theory is involved is that operator algebraists are using ultraproducts.
- Suppose A_i are C*-algebras for all $i \in I$ and that $\mathcal{U}$ is an ultraproduct I. Consider the bounded product

$$\prod^b A_i := \{\bar{a} \in \prod A_i : \lim_{i \to \mathcal{U}} \|a_i\| < \infty\}$$

and the two-sided ideal $c_{\mathcal{U}}$

$$\{\bar{a} \in \prod^b A_i : \lim_{i \to \mathcal{U}} \|a_i\| = 0\}.$$

The ultraproduct, $\prod_{\mathcal{U}} A_i$ is defined as $\prod^b A_i / c_{\mathcal{U}}$.

C*-algebras as metric structures

- We treat C*-algebras as we did Hilbert spaces: there are sorts for each ball of radius $n \in \mathbb{N}$.
- There are inclusion maps between the balls. Additionally there are functions for the restriction of all the operations to the balls. This involves the addition, multiplication, scalar multiplication and the adjoint.
- The metric is given via the operator norm as $\|x - y\|$ on each ball.
- It is routine to check that all of these functions are uniformly continuous (the only issue is multiplication and this holds because we have restricted the norm).
- The sorts are complete since C*-algebras are complete.
- Do we have an elementary class? You would think so since C*-algebras are closed under ultraproducts and subalgebras.

Axioms for C*-algebras

- There are many universal axioms expressing that a C*-algebra is a Banach *-algebra. These involve saying that two terms are equal or that some norm or other is equal to or less than something else.
- For instance, to say $\tau(\bar{x}) = \sigma(\bar{x})$ we need to write $\sup_{\bar{x}} d(\tau(\bar{x}), \sigma(\bar{x}))$ which is awful so we write the first and mean the second.
- For the metric, we have $d(x, 0) = \|x\|$ and $d(x, y) = \|x - y\|$. One can now write out the axioms for a Banach space.
- Include $\|x^*\| = \|x\|$ and the C*-identity, $\|x^*x\| = \|x\|^2$.
- Now comes the fussy bits about using balls: we have

$$\sup_{x \in B_1} \|x\| \leq 1 \text{ and } \sup_{x \in B_n} \min\{1 \dot{-} \|x\|, \inf_{y \in B_1} \|x - y\|\}.$$

- This feels a little awkward since operator algebraists know that C*-algebras are closed under subalgebras and we know that should mean the axioms are universal. They are if you introduce enough terms!

Some results and open questions

- Operator algebraists had already discovered definable sets albeit just for quantifier-free formulas. They called them weakly stable relations.
- The set of self-adjoint elements, projections, partial isometries etc. are all definable sets and this is helpful to know when expressing certain results in continuous logic.
- Coupled with knowing that for a C*-algebra A, $M_n(A)$ can also be viewed as a definable set, the ability to identify projections as a definable set allows one access to the so-called K-theory of A. Understanding what fragment of the K-theory of an algebra is elementary is one open problem in the continuous model theory of C*-algebras.

Some results and open questions

- The use, and usefulness, of ultraproducts in operator algebras made some people ask about how whether there were different ultrapowers of separable C*-algebras. They also asked a subtler question about the so-called relative commutant or central sequence algebra.
- If A is a separable C*-algebra and $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, we consider the subalgebra of $A^{\mathcal{U}}$ called the relative commutant

$$A' \cap A^{\mathcal{U}} = \{b \in A^{\mathcal{U}} : [b, A] = 0\}$$

  where A is identified with its diagonal embedding in $A^{\mathcal{U}}$.

Theorem

For any infinite-dimensional C^*-algebra A, if CH does not hold then there are many non-isomorphic ultrapowers of A via ultrafilters on $\mathbb{N}$. Moreover, there are many non-isomorphic relative commutants as well.

Some results and open questions

- The usefulness of the ultraproduct construction arises from its degree of saturation. Relative commutants are also to some extent saturated - quantifier-free saturated and even more on occasion.
- A general model theory question would be: is there some abstract version of the relative commutant that could be useful in other contexts?