Continuous model theory and the classification problem

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• Basics of C*-algebras
• Reminder about continuous model theory
• Basic model theory of operator algebras
• The Elliott classification problem
• More advanced C*-algebra basics
• A model theory conjecture
C*-algebra basics

Definition

A C*-algebra is a *-subalgebra $A$ of the bounded linear operators $B(H)$ on a complex Hilbert space $H$ which is closed in the operator norm topology. Alternatively, a C*-algebra is a Banach *-algebra $A$ which satisfies the C*-identity $\|a^* a\| = \|a\|^2$ for all $a \in A$.

The first sentence defines a concrete representation of a C*-algebra and the second gives an abstract definition.

Theorem (Gel’fand, Naimark, Sigal)

Every abstract C*-algebra has a concrete representation.

Examples:

- $M_n(\mathbb{C})$; in general, $B(H)$; $C_0(X)$ for any locally compact space $X$ - these form all the commutative C*-algebras.
- C*-algebras are closed under inductive limits where the relevant morphisms are *-homomorphisms.
- C*-algebras are closed under tensor products but ...
Definition
If $A$ is a unital C*-algebra and $a \in A$ then $sp(a)$, the spectrum of $a$, is the set of $\lambda \in \mathbb{C}$ such that $a - \lambda I$ is not invertible.

Theorem (Spectral Theorem)
Suppose $A$ is a unital C*-algebra and $a \in A$ is self-adjoint ($a^* = a$) then $C^*(a)$, the C*-subalgebra of $A$ generated by $a$ and $I$ is isomorphic to $C(sp(a))$ via the map which sends $a$ to the identity and $I$ to 1.

Example: If $A$ is a C*-algebra and $p \in A$, we call $p$ a projection if $p^2 = p = p^*$. 
Claim: For every $\epsilon > 0$ there is a $\delta > 0$ such that if $a$ is self-adjoint and $\|a^2 - a\| < \delta$ then there is a projection $p$ such that $\|p - a\| < \epsilon$. 
A C*-algebra can be thought of as a metric structure by introducing a sort for each ball of operator norm $N \in \mathbb{N}$.

One has function symbols for the sorted operations of $+$, $\cdot$ and $*$ as well as the unary operations of multiplication by $\lambda$ for every $\lambda \in \mathbb{C}$. It is sometimes useful to consider an expanded language in which one has a function symbol for every *-polynomial (again properly sorted).

The only relation symbol is the operator norm $\| \cdot \|$.

The basic formulas of continuous logic which are relevant here are $\| p(\bar{x}) \|$ where $p(\bar{x})$ is a *-polynomial.

Formulas are closed under composition with continuous real-valued functions; moreover, if $\varphi$ is a formula then so is $\sup_{x \in B_N} \varphi$ or $\inf_{x \in B_N} \varphi$. The interpretation of these formulas in a C*-algebra is standard.
The theory of C*-algebras

• Notice the if $A$ is a C*-algebra, $\bar{a} \in A$ and $\varphi$ is a formula then $\varphi^A(\bar{a})$ is a number. In particular, if $\varphi$ is a sentence then $\varphi^A \in \mathbb{R}$.

• $Th(A)$, the theory of an algebra, is the function which to every sentence $\varphi$ assigns $\varphi^A$. A theory is determined by its zero set.

• We say that a class of structures $K$ is elementary if there is a set of sentences $T$ such that $A \in K$ iff $\varphi^A = 0$ for all $\varphi \in T$.

Theorem

The class of C*-algebras is an elementary class. In fact, in the appropriate language it is a universal class.
Ultraproducts

- If $A_i$ for $i \in I$ are C*-algebras and $U$ is an ultrafilter on $I$, one forms the norm ultraproduct as follows:

- Let

$$\ell^{\infty}\left(\prod_{i \in I} A_i\right) = \{ \bar{a} \in \prod_{i \in I} A_i : \text{for some } M, \|a_i\| \leq M \text{ for all } i \in I \}$$

and

$$c_U = \{ \bar{a} \in \ell^{\infty}\left(\prod_{i \in I} A_i\right) : \lim_{i \to U} \|a_i\| = 0 \}$$

- The ultraproduct is then

$$\prod_{i \in I} A_i / U := \ell^{\infty}\left(\prod_{i \in I} A_i\right) / c_U.$$
Definable zero sets

Definition
Suppose that $M$ is a metric structure and $\varphi(\bar{x})$ is a formula. We say that $\varphi$ has a definable zero set if the distance function to the zero set of $\varphi$, $\{\bar{a} \in M : \varphi^M(\bar{a}) = 0\}$, is given by a definable predicate in $M$ i.e. a uniform limit of formulas.

Theorem
For a metric structure $M$ and a formula $\varphi$, the following are equivalent:

- $\varphi$ has a definable zero set.
- The zero set of $\varphi$ can be quantified over.
Stable relations

Definition
In the language of C*-algebras, a formula $\varphi(\bar{x})$, or its zero set, is called a stable relation if for every C*-algebra $A$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\bar{a} \in A$ and $|\varphi(\bar{a})| < \delta$ then there is $\bar{b} \in A$ such that $\varphi(\bar{b}) = 0$ and $\|\bar{a} - \bar{b}\| < \epsilon$.

Lemma
Among C*-algebras, the notions of stable relation and definable zero set are the same.

Examples of stable relations:

- the set of projections.
- the sets of self-adjoint elements, unitary elements ($u^* u = uu^* = 1$), positive elements ($a^* a$); in general, the range of any term.
- the sets of generators for subalgebras isomorphic to $M_n(C)$, for any $n \in \mathbb{N}$ or, in general, any finite-dimensional algebra.
The classification programme for nuclear C*-algebras

The Elliott conjecture

The isomorphism type of a simple, separable, infinite-dimensional, unital nuclear C*-algebra is determined by its K-theory.

- For a C*-algebra $A$, there is an invariant called the Elliott invariant which for the record is defined as:

$$\text{Ell}(A) = ((K_0(A), K_0^+(A), [1_A]), K_1(A), Tr(A), \rho_A)$$

- There are other invariants which come up like KK-theory and the Cuntz semi-group but I won’t focus on them.
Nuclear algebras

**Definition**
A C*-algebra $A$ is called nuclear if for all C*-algebras $B$, $A\overline{\otimes} B$ is uniquely defined.

**Examples:**
- All abelian C*-algebras are nuclear.
- $M_n(\mathbb{C})$ is nuclear but $B(H)$ for an infinite-dimensional Hilbert space $H$ is not nuclear.
- The class of nuclear algebras is closed under tensor products hence $M_n(C(X))$ is nuclear for any compact space $X$.
- The class of nuclear algebras is closed under inductive limits; UHF (uniformly hyperfinite) algebras are limits of matrix algebras; AF (approximately finite dimensional) algebras are limits of finite-dimensional algebras.
- The class of nuclear algebras is not closed under ultraproducts or even ultrapowers.
Nuclear algebras, cont’d

Definition

• A element of a C*-algebra $A$ is said to be positive if it is of the form $a^*a$ for some $a \in A$.

• A linear map $f : A \to B$ is positive if whenever $a \in A$ is positive then so is $f(a)$.

• A linear map $f : A \to B$ is completely positive if the induced map from $M_n(A)$ to $M_n(B)$ is positive for all $n$.

• A map $f$ is contractive if $\|f\| \leq 1$.

Theorem (Stinespring)

For any completely positive map $f : A \to B(H)$ there is a Hilbert space $K$, *-homomorphism $\pi : A \to B(K)$ and $V \in B(K, H)$ such that $f(a) = V\pi(a)V^*$. 
Definition
A C*-algebra $A$ has the contractive positive approximation property (CPAP) if for every $\bar{a} \in A$ and $\epsilon > 0$ there is an $n$ and cpc maps $\sigma : A \to M_n(\mathbb{C})$ and $\tau : M_n(\mathbb{C}) \to A$ such that $\|\bar{a} - \tau(\sigma(\bar{a}))\| < \epsilon$.

Theorem (Choi-Effros, Kirchberg)
A C*-algebra $A$ is nuclear iff it satisfies the CPAP.

Theorem
There are countably many partial types such that a C*-algebra is nuclear iff it omits all of these types.
The definition of $K_0$

Definition
For any C*-algebra $A$, consider the equivalence relation $\sim$ on projections in $A$ given by $p \sim q$ iff there is some $v \in A$, $vpv^* = q$ and $v^*qv = p$.

Consider the (non-unital) *-homomorphism $\Phi_n : M_n(A) \to M_{n+1}(A)$ defined by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

and let $M_\infty = \lim_n M_n(A)$. We should really complete this ...

Let $V(A) = Proj(M_\infty(A))/\sim$.

$V(A)$ has an additive structure defined as follows: if $p, q \in V(A)$ then $p \oplus q$ is

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$
The definition of $K_0$, cont’d

**Definition**

$K_0(A)$ is the Grothendieck group generated from $(V(A), \oplus)$ and $K_0^+(A)$ is the image of $V(A)$ in $K_0(A)$; if $A$ is unital then the constant $[1_A]$ corresponds to the identity in $A$.

**Examples:**

- $K_0(M_n(\mathbb{C}))$ is $(\mathbb{Z}, \mathbb{N}, n)$.
- If $H$ is infinite-dimensional then $K_0(B(H))$ is 0.
- Consider $A = \lim_n M_{2^n}(\mathbb{C})$ where the given morphisms are $M_{2^n}(\mathbb{C}) \hookrightarrow M_{2^{n+1}}(\mathbb{C})$ such that

  $$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

  Then $K_0(A)$ is the dyadic rationals with the unit associated to 1.
Examples of $K_0$, cont’d

- In general, if $A = \lim_k M_{n(k)}$ where $n(k) | n(k + 1)$ for all $k$ and the morphisms are given by diagonal maps

$$a \mapsto \begin{pmatrix} a & 0 \\ a & \ddots \\ 0 & \ddots & a \end{pmatrix}$$

then $K_0(A) = \{m/n : m \in \mathbb{Z} \text{ and } n | n(k) \text{ for some } k\}$. 
• We have already introduced \((K_0(A), K_0^+(A), [1_A])\).
• \(K_1(A) = K_0(C_0((0, 1), A))\).
• \(\text{Tr}(A)\) is the set of traces on \(A\) i.e. all positive linear functionals \(\tau\) on \(A\) such that \(\tau(1) = 1, \tau(x^\ast) = \overline{\tau(x)}\) and \(\tau(xy) = \tau(yx)\).
• \(\rho_A\) is the natural pairing of \(\text{Tr}(A)\) and \(K_0(A)\).
• The form of the Elliott conjecture which states that the Elliott invariant classifies all simple, separable, infinite-dimensional, unital nuclear algebras is false - there are counter-examples of different types with the first ones due to Toms and separately Rørdam.
• A search is on for a new invariant which might classify nuclear algebras.
Theorem (Elliott)

The class of AF algebras can be classified by $K_0$.

Let’s do a special case of this result due to Glimm.

Definition

For a UHF algebra $A = \lim_k M_{n(k)}$, let the $GI(A)$, the generalized integer of $A$ be the function which assigns to every prime $p$ the supremum of all $n$ such that $p^n$ divides $n(k)$ for some $k$; this can be infinite.

Theorem (Glimm)

If $A$ and $B$ are separable, unital UHF algebras then $A \simeq B$ iff $GI(A) = GI(B)$. 

Prototypical example of classification
A proof of Glimm’s theorem

Sketch of proof: One checks that UHF algebras have a unique trace and the values of this trace on a UHF algebra $A$ are of the form $\{k/n : k \in \mathbb{N}, n|GI(A)\}$.

Now if $GI(A) = GI(B)$ then we can arrange in a back and forth fashion that $A = \lim_k M_{n(k)}$ and $B = \lim_k M_{m(k)}$ such that for all $k$, $n(k)|m(k)|n(k+1)$. It is possible then to create a sequence of maps $\varphi_k : M_{n(k)} \to M_{m(k)}$ and $\psi_k : M_{m(k)} \to M_{n(k+1)}$ which additionally have the necessary commutation to make $A$ and $B$ isomorphic.
Model theoretic version of the Elliott conjecture

Simple, separable, infinite-dimensional, unital nuclear algebras are classified by their Elliott invariant and their first order continuous theory.
In the case of a separable, unital UHF algebra $A$, $K_0(A)$ is a rank 1, torsion-free abelian group where we have specified a constant. This is determined by $Gl(A)$ by Glimm’s theorem.

Equivalently, the theory knows the generalized integer for a separable, unital UHF algebra $A = \lim_k M_{m(k)}$. In fact, $M_n$ embeds into $A$ iff $n$ divides $n(k)$ for some $k$.

Round 1 - a draw.
$K_0(A)$ vs. $Th(A)$, round 2

- A classical result of Dixmier which generalizes Glimm’s theorem shows that non-unital separable UHF algebras are classified by $K_0$.
- In this case, $K_0$ is an arbitrary rank 1, torsion-free abelian group.
- The isomorphism relation for such groups is known not to be smooth in the sense of Borel equivalence relations.
- The theory of a C*-algebra is a smooth invariant and so Dixmier’s result shows that $K_0$ and not the theory captures isomorphism at least for non-unital separable UHF algebras.
- Advantage $K_0$ (and descriptive set theory).
The most general counter-examples to the form of the Elliott conjecture which says that $Ell(A)$ is a sufficient invariant are due to Toms, Annals of Math, 2008.

He gave continuum many simple separable nuclear C*-algebras with identical Elliott invariant that were not isomorphic.

He used something called the Cuntz semigroup to show they were not isomorphic and in particular computed a number called the radius of comparison - it was this value that differentiated the algebras.

In joint work with Leonel Robert, we showed that the radius of comparison is known to the theory of an algebra - it is preserved under ultraproducts and elementary submodels.

Advantage $Th(A)$. 
Traces matter

- Nuclear algebras do not form an elementary class but it is interesting to consider the theory of nuclear algebras.
- Question: is every C*-algebra elementarily equivalent to a nuclear algebra?
- No. Let \( A = \prod_{n \in \mathbb{N}} M_n(C)/U \) where \( U \) is a non-principal ultrafilter on \( \mathbb{N} \).
- We need some facts about \( A \): \( A \) has a trace and it is definable say by a formula \( \varphi \).
- Now suppose that \( A \equiv B \) where \( B \) is some simple, separable, nuclear algebra.
Traces matter, cont’d

- But then $\varphi$ would define a trace on $B$ which would mean that the associated von Neumann algebra is the hyperfinite II$_1$ factor $\mathcal{R}$.
- In earlier work with Farah and Sherman we showed that a property identified by von Neumann called property $\Gamma$ for tracial von Neumann algebras was elementary.
- It is known that $\mathcal{R}$ satisfies property $\Gamma$ and that $A$ modulo its trace does not so $A \not\equiv B$.
- Question: what is the theory of the class of nuclear algebras? Is it the theory of C*-algebras?