The transformation T_A

Definition

Suppose that A is an $m \times n$ matrix then T_A is a function with domain R^n and range R^m, usually written

$$T_A : R^n \to R^m$$

defined by: for all $x \in R^n$, $T_A(x) = Ax$.

Theorem

If A is an $m \times n$ matrix, $x, y \in R^n$ and $\lambda \in R$ then

1. $T_A(x + y) = T_A(x) + T_A(y)$ *and*
2. $T_A(\lambda x) = \lambda T_A(x)$

Linear functions

Any function from R^n to R^m which the two properties from the theorem are called linear functions.

Linear functions

- Suppose that $T : R^n \to R^m$ is any linear function.
- Remember that if $x \in R^n$ then $x = \lambda_1 e_1 + \ldots + \lambda_n e_n$ for some $\lambda_1, \ldots, \lambda_n$.
- So $T(x) = \lambda_1 T(e_1) + \ldots + \lambda_n T(e_n)$.
- This says that every linear function is determined by its values on $e_1, \ldots, e_n$.
- Consider the matrix

$$A = (T(e_1)|T(e_2)| \ldots |T(e_n))$$

- We see that $T = T_A$.
- Conclusion: All linear functions from R^n to R^m are of the form T_A for some $m \times n$ matrix A.

Matrix multiplication and composition of functions

- The composition of two linear functions is a linear function.
- If A is $m \times k$ and B is $k \times n$ then we can form $T_A(T_B)$ - the composition of these two functions and it will be a linear function.
- By what was said on the previous slide, this linear function will be T_C for some C; what is C?
- $C = AB$.
- So matrix multiplication is what you get when you compose linear functions.

Determinants

The goal

- To every square matrix A we wish to assign a number called the determinant of A and written $det(A)$.
- We will use this to detect invertibility: $det(A) \neq 0$ iff A is invertible.

Definition

For a square matrix A, we define

- The *ij* minor of A, M_{ij}, is the determinant of the square matrix obtained from A by deleting the i^{th} row and j^{th} column of A.
- The *ij* cofactor of A is $(-1)^{i+j}M_{ij}$.

Cofactor expansion

Definition

Suppose that A is an $n \times n$ square matrix.

- The cofactor expansion along the i^{th} row is

$$a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

- The cofactor expansion along the j^{th} column is

$$a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

Theorem

Any cofactor expansion of a square matrix A, along any row or any column, always yields the same number and we call that number the determinant of A.

Some easy facts

- If A is a triangular matrix then $det(A)$ is the product of the diagonal entries.
- If a square matrix has a row or column which is entirely zero then its determinant is 0.
- If A is a square matrix then $det(A) = det(A^T)$.
- If B is a square matrix obtained by multiplying a row or column of A by k then $det(B) = kdet(A)$.
- If B is obtained from A by exchanging two rows then $det(B) = -det(A)$.