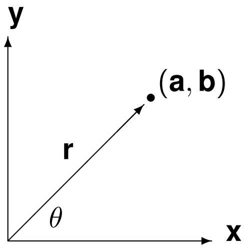
The complex plane



- $r = \sqrt{a^2 + b^2}$; this is called the modulus of the complex number $z = a + bi$ and written $|z|$.
- We saw that $z \cdot \bar{z} = |z|^2$.
- θ is an argument for $a + bi$ and is only determined up to multiples of 2π.
- $a = r\cos(\theta)$ and $b = r\sin(\theta)$ so $z = r(\cos(\theta) + i\sin(\theta))$.

A beautiful formula and a caveat going forward

- Exponentiation can be defined on complex numbers as follows:
$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$
- Consider then that if $\theta = \pi$ then
$$e^{i\pi} + 1 = 0$$

Linear algebra over the complex numbers

Everything we have done in the course to date - linear systems, matrices, determinants, eigenvalues, diagonalization - all goes through **unchanged** if we are using complex numbers instead of real numbers. Going forward it will be considered fair to have complex entries in matrices and to consider complex roots of polynomials.

Definition

- $\mathbb{R}^n$ or real n-space is the collection of all n-tuples $v = (v_1, v_2, \ldots, v_n)$ where $v_1, \ldots, v_n \in \mathbb{R}$. We refer to elements of $\mathbb{R}^n$ as vectors.
- We define addition between vectors in $\mathbb{R}^n$ as follows:

$$(u_1, \ldots, u_n) + (v_1, \ldots, v_n) = (u_1 + v_1, \ldots, u_n + v_n)$$

- Scalar multiples of vectors in $\mathbb{R}^n$ is defined, for $r \in \mathbb{R}$ by

$$r(v_1, \ldots, v_n) = (rv_1, \ldots, rv_n)$$

Properties of $\mathbb{R}^n$

- The vector $0 = (0, \ldots, 0)$ is called the zero vector.
- Lots of interesting properties involving +, scalar multiplication and the zero vector contained in Theorem 3.1.1.
- If $k_1, \ldots, k_m$ are numbers and $v_1, \ldots, v_m$ are vectors then $k_1 v_1 + \ldots k_m v_m$ is a vector and is called a linear combination of $v_1, \ldots, v_m$.
- The norm of a vector $v = (v_1, \ldots, v_n)$, written $||v||$, is $\sqrt{v_1^2 + \ldots + v_n^2}$ and there is a corresponding notion of distance $d(u, v) = ||u - v||$.

Properties of $\mathbb{R}^n$, cont'd

Theorem (3.2.1)

For $v \in \mathbb{R}^n$,

- $||v|| \geq 0$
- $||v|| = 0$ *iff* $v = 0$
- $||kv|| = |k|||v||$

The geometry of $\mathbb{R}^n$

Definition

For two vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ we define the dot product

$$(u_1, \ldots, u_n) \cdot (v_1, \ldots, v_n) = u_1 v_1 + \ldots + u_n v_n$$

and the angle θ between non-zero vectors u and v by

$$\cos(\theta) = \frac{u \cdot v}{||u||||v||}$$

The geometry of $\mathbb{R}^n$, cont'd

- Theorems 3.2.2 and 3.2.3 have many valuable parts; I draw your attention to the part which says: for a vector v, $v \cdot v \geq 0$ and $v \cdot v = 0$ iff $v = 0$.
- Theorem 3.2.4, the Cauchy-Schwartz inequality says that for all vectors $u, v \in \mathbb{R}^n$, $|u \cdot v| \leq ||u||||v||$.
- This allows us to prove the triangle inequality: for all vectors $u, v \in \mathbb{R}^n$, $||u + v|| \leq ||u|| + ||v||$.
- Theorems 3.2.6 and 3.2.7 express in n-space two geometric results: for all vectors $u, v \in \mathbb{R}^n$,

$$||u + v||^2 + ||u - v||^2 = 2(||u||^2 + ||v||^2) \text{ and}$$

$$u \cdot v = \frac{1}{4}||u + v||^2 - \frac{1}{4}||u - v||^2$$