Echelon forms

A matrix is in **row echelon form** if

1. If a row isn't entirely zeroes then the left most non-zero entry is a 1; we call this 1 the row's leading 1.
2. All zero rows are grouped contiguously at the bottom of the matrix.
3. In any two consecutive rows which are both non-zero, the leading 1 of the upper row is to the left of the leading 1 of the lower row.
4. If additionally, any column which contains a leading 1 has zeroes elsewhere then we say the matrix is in **reduced row echelon form**.

Gaussian and Gauss-Jordan elimination

1. Given the augmented matrix for a system of linear equations, find the left most column which is not all zero.
2. Interchange rows so that the non-zero entry from the previous step is the top row.
3. If the non-zero entry you have found is a then divide the top row by a so that its left most entry is 1.
4. Work down row by row adding multiples of the first row to each row to guarantee that all entries below this 1 are zero.
5. Now ignore the top row and repeat the process with the rows you have remaining until there are no more rows left; this gets the matrix into row echelon form.
6. For reduced row echelon form, start with the right most leading 1 and working left, add suitable multiples of that row to the rows above to guarantee that all entries above leading 1's are 0.

Homogeneous systems

- We say that a system of linear equations is a homogeneous system if all the constants in the equations (the right-hand sides) are 0.
- Any homogeneous system of linear equations has at least one solution.
- Theorem: Any homogeneous system with more unknowns than equations has infinitely many solutions.

Matrices

A rectangular array of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

is called a matrix.
The matrix A has m rows and n columns; we say that A is an $m \times n$ matrix. The entry in the i^{th} row and j^{th} column is a_{ij}; also written $(A)_{ij}$.

We say two matrices A and B are equal if they have the same number of rows and columns and for all relevant i and j, $(A)_{ij} = (B)_{ij}$.

Basic algebraic operations

- If A and B are $m \times n$ matrices then we define $A + B$ to be the $m \times n$ matrix whose ij entries are $(A)_{ij} + (B)_{ij}$.
- If A is an $m \times n$ matrix and c is any number then cA is the $m \times n$ matrix whose ij entries are $c(A)_{ij}$.

Matrix multiplication

- If A is an $m \times k$ matrix and B is a $k \times n$ matrix then we can multiply A by B forming AB (the order is important). AB is an $m \times n$ matrix.
- For i and j such that $1 \leq i \leq m$ and $1 \leq j \leq n$ then we need to specify the ij entry of AB.
- Suppose the i^{th} row of A and j^{th} column of B are

$$\begin{pmatrix} a_{i1} & a_{i2} & \ldots & a_{ik} \end{pmatrix} \text{ and } \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{kj} \end{pmatrix}$$

Then the ij entry of AB is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}$$

Back to linear systems

- Suppose we have the linear system

$$
\begin{array}{cccc}
a_{11}x_1 + a_{12}x_2 + & \dots & + a_{1n}x_n & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + & \dots & + a_{2n}x_n & = & b_2 \\
\vdots & \ddots & \vdots & & \\
a_{m1}x_1 + a_{m2}x_2 + & \dots & + a_{mn}x_n & = & b_m
\end{array}
$$

- Let A, x and b be the matrices

$$
\begin{pmatrix}
a_{11} & a_{12} & \dots & a_{1n} \\
a_{21} & a_{22} & \dots & a_{2n} \\
\vdots & \ddots & \vdots & \\
a_{m1} & a_{m2} & \dots & a_{mn}
\end{pmatrix},
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\text{ and }
\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}
$$

- Then the linear system can be written using matrices as $Ax = b$.